UIUC Mock Putnam Exam 3/2003
Elementary Version

11-19-03

Problem 1. Let
$$f(x) = \frac{1}{1-x}.$$
Let $f_1(x) = f(x)$ and for each $n = 2, 3, \ldots$, let $f_n(x) = f(f_{n-1}(x))$. What is the value of $f_{2003}(2003)$?

Solution. $f_{2003}(2003) = 1 - \frac{1}{2003}$.

Observe that
$$f_1(x) = \frac{1}{1-x},$$
$$f_2(x) = \frac{1}{1 - \frac{1}{1-x}} = \frac{1-x}{-x} = 1 - \frac{1}{x}$$
and
$$f_3(x) = \frac{1}{1 - (1 - \frac{1}{x})} = x.$$
Thus $f_4(x) = f_1(x) = 1/(1-x)$, $f_5(x) = f_2(x) = 1 - (1/x)$, etc. The functions $f_n(x)$ repeat with period three. Since 2001 is a multiple of three, $f_{2001}(x) = x$ and $f_{2003}(x) = f_2(x) = 1 - (1/x)$.

Problem 2. Given $x_0 = 0$, define $x_{k+1} = \frac{x_k^2 - 2}{2x_k - 3}$. Determine if the sequence (x_n) is convergent and if it is, find the limit.

Solution. The sequence in question converges to one. In fact,
$$x_k = \frac{2^{2^k} - 2}{2^{2^k} - 1}, \qquad k = 0, 1, 2, \ldots \tag{1}$$

We prove (1) by induction. The case $k = 0$ is clearly true. Assume that (1) holds for some index k and compute

$$\begin{aligned}\frac{x_k^2 - 2}{2x_k - 3} &= \frac{\left(\frac{2^{2^k}-2}{2^{2^k}-1}\right)^2 - 2}{2\left(\frac{2^{2^k}-2}{2^{2^k}-1}\right) - 3} \\ &= \frac{(2^{2^k} - 2)^2 - 2(2^{2^k} - 1)^2}{(2^{2^k} - 1)(2(2^{2^k} - 2) - 3(2^{2^k} - 1))} \\ &= \frac{2 - 2^{2^{k+1}}}{(2^{2^k} - 1)(-1 - 2^{2^k})} = \frac{2^{2^{k+1}} - 2}{2^{2^{k+1}} - 1} = x_{k+1}.\end{aligned}$$

Problem 3. Show that there is a multiple of 2003 which contains all ten digits.

Solution. For integers a, b, c, d with $0 \leq a, b, c, d \leq 9$, consider the number whose base ten expansion is $1234567890abcd$. There are $10,000$ consecutive numbers of this type; at least one of these numbers must be a multiple of 2003.

Problem 4. Show that among any 101 points inside a square whose side has length 1, there exist two points whose distance is at most $\frac{\sqrt{2}}{10}$.

Solution. Divide the square into 100 subsquares whose sides are all of length 1/10 as indicated in the figure.

By the Pigeonhole Principle, at least one of these subsquares must contain two of the points in question. Denoting these points by $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ and observing that $|x_2 - x_1| \leq 0.1$, $|y_2 - y_1| \leq 0.1$, we see that

$$||p_2 - p_1|| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \leq \sqrt{0.02} = \frac{\sqrt{2}}{10}$$

as desired.

Problem 5. Let (a_n) be a bounded sequence of integers which satisfies the recurrence condition

$$a_n = \frac{a_{n-1} + a_{n-2} + a_{n-3}a_{n-4}}{a_{n-1}a_{n-2} + a_{n-3} + a_{n-4}}. \tag{2}$$

Show that the sequence is eventually periodic.

Solution. Let (a_n) be a bounded sequence of integers satisfying (2). Choose M so large that $-M \leq a_n \leq M$ for all n, and consider quadruples of the form $q_n := (a_n, a_{n-1}, a_{n-2}, a_{n-3})$. Because of the bound on the values of a_n, there are at most $N := (2M + 1)^4$ such quadruples. By the Pigeonhole Principle, within any collection of $N + 1$ quadruples

$$q_n, q_{n+1}, \ldots, q_{n+N}$$

there must be two identical quadruples q_{n+j} and q_{n+k}, $0 \leq j, k \leq N$, $j \neq k$. It follows that the sequence (a_{n+i}), $i = 1, 2, \ldots$, is periodic with period $|k-j|$.

Observe that bounded sequences of integers (a_n) which satisfy (2) do exist. For example, the constant sequence $a_n = 1$, $n = 1, 2, 3, \ldots$, satisfies (2).