UIUC Mock Putnam Exam 1/2004
Solutions

Problem 1. Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Evaluate the series

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}}$$

Solution. Writing

$$\frac{1}{F_n F_{n+2}} = \frac{1}{F_{n+2} - F_n}\left(\frac{1}{F_n} - \frac{1}{F_{n+2}}\right) = \frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1} F_{n+2}},$$

and substituting this expression formally into the given series would give $1/(F_1 F_2) = 1$ as the sum. However, while the answer turns out to be correct, this argument is not rigorous. To make it rigorous, we need to work with partial sums. Using the above identity we see that the n-th partial sum of the given series is $1 - 1/(F_{n+1}F_{n+2})$, and since this tends to 1 as $n \to \infty$, the sum of the given series is indeed 1.

Problem 2. Let $\{c_n\}_{n\geq 1}$ be a sequence of positive numbers, and suppose that, for any sequence $\{d_n\}_{n\geq 1}$ of positive numbers with $\lim_{n\to\infty} d_n = 0$, the series $\sum_{n=1}^{\infty} c_n d_n$ converges. Show that the series $\sum_{n=1}^{\infty} c_n$ converges as well.

Solution. We argue by contradiction. Given a sequence $\{c_n\}$ of positive numbers such that $\sum_{n=1}^{\infty} c_n$ diverges, we will construct a sequence $\{d_n\}$ of positive numbers tending to 0, such that $\sum_{n=1}^{\infty} c_n d_n$ diverges. This will prove the assertion.

So suppose $c_n > 0$ and $\sum_{n=1}^{\infty} c_n$ diverges. Then the partial sums $\sum_{n=1}^{N} c_n$ tend to infinity as $N \to \infty$. It follows that we can break up the series into "junks" whose sizes tend to infinity; specifically, there exists a sequence $1 = N_0 < N_1 < \cdots$ of integers such that if $S_i = \sum_{n=N_i}^{N_{i+1}-1} c_n$, then $(*)$ $S_i \to \infty$ as $i \to \infty$. Now define $d_n = 1/S_i$ if $N_i \leq n < N_{i+1}$. Then $(*)$ implies that $d_n \to 0$ as $n \to \infty$, so the sequence $\{d_n\}$ satisfies the conditions stated in the problem. On the other hand, by construction we have $\sum_{n=N_i}^{N_{i+1}-1} c_n d_n = 1$ for each i, so $\sum_{n=1}^{\infty} c_n d_n$ diverges. This is the desired contradiction.

Problem 3. Let $a_1 = a_2 = a_3 = 1$, and for $n \geq 4$ define a_n recursively by

$$a_n = \frac{1 + a_{n-1}a_{n-2}}{a_{n-3}}.$$

Show that a_n is an integer for all n.

Solution. Computing the first few terms of the sequence, we obtain $1, 1, 1, 2, 3, 7, 11, 26, 41, 97, 153, \ldots$. Forming the difference sequence (a standard trick!) gives $0, 0, 1, 1, 4, 4, 15, 15, 56, 56, \ldots$. This suggests that the differences $d_n = a_{n+1} - a_n$ satisfy $d_{2n} = d_{2n-1}$ and $d_{2n} = 4d_{2n-2} - d_{2n-4}$. The latter relations can be verified (with some effort) by induction, and since these relations clearly imply that the terms d_n, and hence also a_n, are all integers, the claim follows.

Alternative solution: Let $q_n = (a_{n+1} + a_{n-1})/a_n$ for $n \geq 2$. The given recurrence implies (by an easy induction) that the terms a_n are all positive, so q_n is well-defined. Also, the recurrence for a_n implies

$$\begin{aligned} 1 + a_n a_{n+1} &= a_{n-1}a_{n+2}, \\ a_n a_{n+3} &= 1 + a_{n+1}a_{n+2}. \end{aligned}$$

Adding these relations gives

$$a_n(a_{n+1} + a_{n+3}) = a_{n+2}(a_{n-1} + a_{n+1}).$$

It follows that $q_{n+2} = q_n$ for all $n \geq 2$. Hence $q_n = q_2 = (a_1 + a_3)/a_2 = 2$ for all even $n \geq 2$ and $q_n = q_3 = (a_2 + a_4)/a_3 = 3$ for all odd $n \geq 3$. In particular, q_n is an integer for all n. Since $a_{n+1} = q_n a_n + a_{n-1}$, it follows by induction that a_n is an integer for all n.

Problem 4. $2n$ points are drawn on the circumference of a circle. In how many ways can these points be joined in pairs by n chords which do not intersect within the circle?

Solution. This is a rather well-known problem in combinatorics and can be found in many books on combinatorics. The answer is given by the so-called Catalan number, $(1/(n+1))\binom{2n}{n}$. Here is an outline of the argument:

Let a_n denote the number in question. Pick one of the $2n$ points, say P. In any admissible pairing of points, this point must be joined by a chord to exactly one other point, say Q. Fix Q. Then all chords other than PQ must lie completely either to the left or to the right of the chord PQ. Since the points are connected by chords in pairs, it follows that either side of the chord must contain an even number of points, say $2k$ points to the left, and $2(n-k-1)$ points to the right (since there are $2n-2$ points other than P and Q) and that the number of admissible configurations of chords to the left of the chord PQ is a_k, and that to the right is a_{n-k-1}. Summing over all possible choices of Q (corresponding to the values $k = 0, 1, \ldots, 2n-1$), we get the recurrence

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-k-1} \quad (n \geq 1), \tag{1}$$

with initial condition $a_0 = a_1 = 1$.

To solve this recurrence, we introduce the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$. The recurrence (1) then translates into the generating function identity

$$f(x) - 1 = \frac{1}{x} f(x)^2.$$

This is a quadratic equation in $f(x)$, with two solutions

$$f_{\pm}(x) = \frac{1 \pm (1-4x)^{1/2}}{2x}$$

The solution $f_+(x)$ can be discarded since it is discontinuous at $x = 0$, so $f(x)$ must be equal to $f_-(x)$.

Having determined $f(x)$, we use the binomial series $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ to expand $f(x)$ into a power series and read off the n-th coefficient to get a_n. We have

$$f(x) = -\frac{1}{2x} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n x^n = \frac{1}{2} \sum_{n=0}^{\infty} \binom{1/2}{n+1} 4^{n+1} (-1)^n x^n,$$

so the coefficient of x^n is $a_n = \binom{1/2}{n+1} 2^{2n+1} (-1)^n$. Now an easy exercise in manipulating binomial coefficients shows that this is the same as $a_n = \frac{1}{n+1}\binom{2n}{n}$.