UIUC Mock Putnam Exam 2/2004

Problem 1. Show that, for all integers $n \geq 7$,

$$\sqrt{n}^{\sqrt{n+1}} > \sqrt{n+1}^{\sqrt{n}}.$$

Problem 2. Let $f(n)$ denote the number of **ordered** tuples $(r_1, \ldots, r_k)$ of positive integers with $r_1 + \cdots + r_k = n$. For example, $f(3) = 4$, since 3 has four representations of this type: $3 = 3$, $3 = 1 + 2$, $3 = 2 + 1$, $3 = 1 + 1 + 1$. Find and prove a general formula for $f(n)$.

Problem 3. Consider a matrix consisting of infinitely many rows and finitely many columns defined as follows. The top row consists of an arbitrary finite sequence of integers, not necessarily distinct. Given a row with entries $a_1, a_2, \ldots, a_n$, the i-th entry in the following row is defined as the number of occurrences of the number a_i among the entries $a_1, a_2, \ldots, a_n$. For example, if the given row has entries $1, 2, 1, 3$, the following row has entries $2, 1, 2, 1$. Prove that, from some point onwards, all rows must be identical.

Problem 4. Let $f(x)$ be a polynomial satisfying $f(x) \geq 0$ for all x, and let $F(x) = \sum_{i=0}^{\infty} f^{(i)}(x)$, where $f^{(i)}$ is the i-th derivative of f. (Note that since $f(x)$ is a polynomial, all derivatives $f^{(i)}(x)$ of sufficiently large order are identically zero, so the series is a finite series.) Show that $F(x) \geq 0$ for all $x \geq 0$.