UIUC Mock Putnam Exam 2/2004
Solutions

Problem 1. Show that, for all integers $n \geq 7$,

$$\sqrt{n}^{\sqrt{n+1}} > \sqrt{n+1}^{\sqrt{n}}.$$

Solution. Taking logarithms, we see that the asserted inequality is equivalent to

$$\sqrt{n+1} \log \sqrt{n} > \sqrt{n} \log \sqrt{n+1},$$

which in turn can be written as

$$\frac{\log \sqrt{n}}{\sqrt{n}} > \frac{\log \sqrt{n+1}}{\sqrt{n+1}}$$

To prove the latter inequality it suffices to check that the function $f(x) = \log x/x$ (where log is the natural logarithm) is decreasing for $x \geq \sqrt{7}$. Now, $f'(x) = (1 - \log x)x^{-2}$, which is negative for $x > e$, and since $e = 2.71 \ldots > \sqrt{7} = 2.6 \ldots$, the desired result follows.

Problem 2. Let $f(n)$ denote the number of **ordered** tuples $(r_1, \ldots, r_k)$ of positive integers with $r_1 + \cdots + r_k = n$. For example, $f(3) = 4$, since 3 has four representations of this type: $3 = 3$, $3 = 1 + 2$, $3 = 2 + 1$, $3 = 1 + 1 + 1$. Find and prove a general formula for $f(n)$.

Solution. We will show by induction that $f(n) = 2^{n-1}$. For $n = 1$ there is only the one representation $1 = 1$, so $(*)$ holds in this case. Now suppose that $n \geq 2$ and that desired formula holds for $1, 2 \ldots, n-1$. We need to count representations of n in the form $n = r_1 + r_2 + \cdots + r_k$ with positive integers r_i. First note that there is exactly one such representation consisting of only one term, namely $n = n$. Next, group the representations involving more than one term according to the value of the first term, r_1. Clearly, r_1 can take any of the values $1, 2, \ldots, n-1$. For a given value of r_1, say t, the representations of n in the above form in which $r_1 = t$ and $k \geq 2$ are in one-to-one correspondence with the (unrestricted) representations of $n-t$ of the required form. By the induction hypothesis, there are $f(n-t) = 2^{n-t-1}$ such representations. Hence the total number of representations of n in the required form is

$$1 + \sum_{t=1}^{n-1} 2^{n-t-1} = 2^{n-1},$$

so $f(n) = 2^{n-1}$. This completes the induction.

Problem 3. Consider a matrix consisting of infinitely many rows and finitely many columns defined as follows. The top row consists of an arbitrary finite sequence of integers, not necessarily distinct. Given a row with entries $a_1, a_2, \ldots, a_n$, the i-th entry in the following row is defined as the number of occurrences of the number a_i among the entries $a_1, a_2, \ldots, a_n$. For example, if the given row has entries $1, 2, 1, 3$, the following row has entries $2, 1, 2, 1$. Prove that, from some point onwards, all rows must be identical.

Solution. The proof follows from the following observations:

(1) First, from the second row onwards, any entry k in a given row of that matrix must appear at least k times in that row. This is because, by the construction of the matrix, k counts the number of occurrences of a given entry in the previous row, and for every occurrence of that entry in the previous row the corresponding entry in the given row is k.

(2) As a consequence of (1), from the second row onwards, given any entry, say k, the entry in the same column in the next row must be at least equal to k. Hence, in any given column the entries in that column, from the second term onwards, form a non-decreasing sequence.

(3) If n is the number of columns in the matrix (which, by hypothesis, is finite), then all entries in the matrix from the second column onwards are bounded from above by n.

(4) From (2) and (3), we see that the entries in each column from the second row onwards form a non-decreasing and bounded sequence, and therefore converge to a limit. Since the entries are all integers, this means that the terms of each column sequence must be equal from some point onwards. Since there are finitely many columns, this implies that from some point onwards, all column sequences are stationary, and therefore that the rows beyond that point are identical.

Problem 4. Let $f(x)$ be a polynomial satisfying $f(x) \geq 0$ for all x, and let $F(x) = \sum_{i=0}^{\infty} f^{(i)}(x)$, where $f^{(i)}$ is the i-th derivative of f. (Note that since $f(x)$ is a polynomial, all derivatives $f^{(i)}(x)$ of sufficiently large order are identically zero, so the series is a finite series.) Show that $F(x) \geq 0$ for all $x \geq 0$.

Solution. Let $f(x) = \sum_{k=0}^{n} a_k x^k$ with $a_n \neq 0$. Then $F(x)$ is a polynomial of order n consisting of the term $a_n x^n$ plus terms involving lower powers of x. Thus, as $x \to \infty$, $F(x) \sim f(x) \sim a_n x^n$, and since $f(x)$ is nonnegative for $x \geq 0$, it follows that a_n is positive and that both $f(x)$ and $F(x)$ tend to infinity as $x \to \infty$. In particular, $F(x)$ must be bounded from below on the positive real axis. Therefore $F(x)$ attains a minimum value at some point $x_0 \geq 0$, and it suffices to show that $F(x_0) \geq 0$. If $x_0 > 0$ then F has a local minimum at x_0, and therefore $F'(x_0) = 0$. But the definition of $F(x)$ implies that $F'(x) = F(x) - f(x)$, so we get $F(x_0) = F'(x_0) + f(x_0) = f(x_0) \geq 0$ as desired, since $f(x_0) \geq 0$ by hypothesis. On the other hand, if $x_0 = 0$, i.e., if the minimal value of $F(x)$ for $x \geq 0$ occurs at 0, then we must have $F'(0) \geq 0$, and so we obtain again $F(0) = F'(0) + f(0) \geq f(0) \geq 0$.