UIUC Mock Putnam Exam 3/2004
Solutions

Problem 1. Let $f(x) = x^3 e^{x^2}(1-x^2)^{-1}$. Find $f^{(7)}(0)$, the 7th derivative of f at 0. (No brute force calculations!)

Solution. Since $f^{(n)}(0) = a_n n!$, where a_n is the nth Taylor series coefficient of f, it suffices to expand f into a Taylor series and read off the appropriate coefficient.

$$f(x) = x^3 \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right)$$

The coefficient of x^7 here is $1/2! + 1/1! + 1 = 5/2$, so $f^{(7)}(0) = 7!(5/2)$.

Problem 2. Does there exist a multiple of 2004 whose decimal representation involves only a single digit? Explain!

Solution. The answer is yes; specifically, we will show that there exists a multiple of 2004 of the form $4444\ldots4$.

Given a digit $d \in \{1, 2, \ldots, 9\}$, let $N_{d,k}$ be the number whose decimal representation consists of k digits d. Note that

$$N_{d,k} = d\sum_{i=0}^{k-1} 10^i = \frac{d(10^k-1)}{9}.$$

Thus, a given positive integer m has a multiple of this form if and only if the congruence $(*)$ $d(10^k - 1) \equiv 0 \bmod 9m$ has a solution k. We apply this with $d = 4$ and $m = 2004$. Then $(*)$ is equivalent to $10^k - 1 \equiv 0 \bmod 9(2004/4) = 4509$. The latter congruence is equivalent to $(**)$ $10^k \equiv 1 \bmod 4509$. Now, since 10 and 4509 are relatively prime, by Euler's generalization of Fermat's theorem, $(**)$ holds with $k = \phi(4509)$, where $\phi(n)$ is the Euler Phi function, as claimed. (Although not required for the problem, the above value k can be explicitly computed by factoring 4509 and using the multiplicativity of the

Euler function: $4509 = 3^3 \cdot 167$, so $\phi(4509) = (3^3 - 3^2)(167 - 1) = 18 \cdot 166 = 2988$. It follows that a string of 2988 digits 4 is a multiple of 2004.)

Problem 3. Evaluate the sum $\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k$.

Solution. First note that without the factor $(-1)^k$ the given sum would be equal to $\binom{2n}{n}$, by a standard (and not too hard to prove) binomial identity. With this factor the evaluation becomes a bit more complicated. We begin by writing one of the factors $\binom{n}{k}$ as $\binom{n}{n-k}$, so that

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n}{n-k}.$$

The latter sum is the coefficient of x^n in the product

$$\left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k \right) \left(\sum_{k=0}^{n} \binom{n}{m} x^m \right),$$

which, by the binomial theorem is equal to

$$(1-x)^n (1+x)^n = (1-x^2)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2k}.$$

The coefficient of x^n in the last expression, and thus, $f(n)$, is 0 if n is odd, and $(-1)^{n/2} \binom{n}{n/2}$ if n is even.

Problem 4. Let f be a real-valued continuously differentiable function on the interval $[0,1]$ with $\int_0^1 f(x) = 0$. Show that

$$\int_0^1 f(x)^2 dx \le \frac{1}{2} \int_0^1 |f'(x)| dx \int_0^1 |f(x)| dx.$$

Solution. Let $F(x) = \int_0^x f(y) dy$. Note that since $F(0) = 0$ and, by hypothesis, $F(1) = \int_0^1 f(x) dx = 0$, we have, for any $y \in [0,1]$,

$$\begin{aligned} |F(y)| &\le \min(|F(1) - F(y)|, |F(y)|) \\ &\le \min\left(\int_y^1 |f(x)| dx, \int_0^1 |f(x)| dx \right) \\ &\le \frac{1}{2} \int_0^1 |f(x)| dx. \end{aligned}$$

Writing $f(x)^2 = f(x)F'(x)$ and integrating by parts, we get

$$\int_0^1 f(x)^2 dx = f(x)F(x)\Big|_0^1 - \int_0^1 F(x)f'(x)dx$$
$$\leq \int_0^1 |F(x)|f'(x)|dx \leq \max_{y\in[0,1]} |F(y)| \int_0^1 |f'(x)|dx.$$

Using the above bound for $|F(y)$ then gives the asserted inequality.