U OF I MOCK PUTNAM EXAM
SEPT. 29, 2008

1. A sequence $a_0, a_1, a_2 \dots$ of real numbers is defined recursively by
$$a_0 = 1, \quad a_{n+1} = \frac{a_n}{1 + na_n} \quad (n = 0, 1, 2, \dots).$$
Find a general formula for a_n.

2. Consider a 7×7 checkerboard with the squares at the four corners removed (so that the remaining board has 45 squares). Is it possible to cover this board with 1×3 tiles so that no two tiles overlap? Explain!

3. Let f be a function on $[0, 2\pi]$ with continuous first and second derivatives and such that $f''(x) > 0$ for $0 < x < 2\pi$. Show that the integral $\int_0^{2\pi} f(x) \cos x \, dx$ is positive.

4. Given a nonnegative integer b, call a nonnegative integer $a \le b$ a *subordinate* of b if each decimal digit of a is at most equal to the decimal digit of b in the same position (counted from the right). For example, 1329 and 316 are subordinates of 1729, but 1338 is not since the second-last digit of 1338 is greater than the corresponding digit in 1729. Let $f(b)$ denote the number of subordinates of b. For example, $f(13) = 8$, since 13 has exactly 8 subordinates: $13, 12, 11, 10, 3, 2, 1, 0$. Find a simple formula for the sum
$$S(n) = \sum_{0 \le b < 10^n} f(b).$$

5. Let $a_1, a_2, \dots, a_{65}$ be positive integers, none of which has a prime factor greater than 13. Prove that, for some i, j with $i \ne j$, the product $a_i a_j$ is a perfect square.

6. Let n be an *even* positive integer, and let S_n denote the set of all permutations of $\{1, 2, \dots, n\}$. Given two permutations $\sigma_1, \sigma_2 \in S_n$, define their distance $d(\sigma_1, \sigma_2)$ by
$$d(\sigma_1, \sigma_2) = \sum_{k=1}^{n} |\sigma_1(k) - \sigma_2(k)|$$
Determine, with proof, the maximal distance between two permutations in S_n, i.e., determine the exact value of $\max_{\sigma_1, \sigma_2 \in S_n} d(\sigma_1, \sigma_2)$.

[Solutions at `http://www.math.uiuc.edu/contests.html`]