U OF I MOCK PUTNAM EXAM
SEPT. 29, 2008
Solutions

1. A sequence $a_0, a_1, a_2 \dots$ of real numbers is defined recursively by

$$a_0 = 1, \quad a_{n+1} = \frac{a_n}{1 + na_n} \quad (n = 0, 1, 2, \dots).$$

Find a general formula for a_n.

Solution. Set $b_n = 1/a_n$. The given recurrence then takes the form

$$b_0 = 1, \quad b_{n+1} = b_n + n \quad (n = 0, 1, 2, \dots).$$

Iterating this recurrence we obtain, for $n = 1, 2, \dots,$

$$b_n = b_{n-1} + (n-1) = \cdots = b_0 + \sum_{k=0}^{n-1} k = 1 + \frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n + 1.$$

Hence

$$a_n = \frac{1}{b_n} = \left(\frac{1}{2}n^2 - \frac{1}{2}n + 1\right)^{-1} \quad (n = 1, 2, \dots).$$

2. Consider a 7×7 checkerboard with the squares at the four corners removed (so that the remaining board has 45 squares). Is it possible to cover this board with 1×3 tiles so that no two tiles overlap? Explain!

Solution. We will show that no such cover exists. "Color" the board in checkerboard-like fashion, but using three colors instead of two. With the colors labeled $1, 2, 3$, the colored board now looks like this:

	1	2	3	1	2	
1	2	3	1	2	3	1
2	3	1	2	3	1	2
3	1	2	3	1	2	3
1	2	3	1	2	3	1
2	3	1	2	3	1	2
	1	2	3	1	2	

Note that any three squares that are horizontally or vertically adjacent are colored by all three colors $1, 2, 3$, in some order. Thus, any 3×1 tile in a cover of the board must cover exactly one square of each color. Since we need 15 tiles to cover the 45 squares without overlaps, it follows that a cover with these tiles would cover exactly 15 squares of each of the colors $1, 2, 3$, so the board would need to have exactly 15 squares of each color. However, a simple count of the occurrences of these colors in the above coloring shows that there are 16 squares of color 1, 16 squares of color 2, and 13 squares of color 3. Hence such a cover is impossible.

3. Let f be a function on $[0, 2\pi]$ with continuous first and second derivatives and such that $f''(x) > 0$ for $0 < x < 2\pi$. Show that the integral $\int_0^{2\pi} f(x) \cos x \, dx$ is positive.

   Solution. Integrating by parts twice, we obtain

$$\begin{aligned}
\int_0^{2\pi} f(x) \cos x &= (\sin x) f(x) \Big|_0^{2\pi} - \int_0^{2\pi} f'(x) \sin x \, dx \\
&= (\cos x) f'(x) \Big|_0^{2\pi} - \int_0^{2\pi} f''(x) \cos x \, dx \\
&= f'(2\pi) - f'(0) - \int_0^{2\pi} f''(x) \cos x \, dx \\
&= \int_0^{2\pi} f''(x)(1 - \cos x) dx.
\end{aligned}$$

   The last integral is strictly positive since, for $0 < x < 2\pi$, $f''(x)(1 - \cos x) > 0$, by the assumption $f''(x) > 0$.

4. Given a nonnegative integer b, call a nonnegative integer $a \le b$ a *subordinate* of b if each decimal digit of a is at most equal to the decimal digit of b in the same position (counted from the right). For example, 1329 and 316 are subordinates of 1729, but 1338 is not since the second-last digit of 1338 is greater than the corresponding digit in 1729. Let $f(b)$ denote the number of subordinates of b. For example, $f(13) = 8$, since 13 has exactly 8 subordinates: $13, 12, 11, 10, 3, 2, 1, 0$. Find a simple formula for the sum

$$S(n) = \sum_{0 \le b < 10^n} f(b).$$

   Solution. We will show that $S(n) = 55^n$. First note that since $f(b)$ counts the number of pairs (a, b) such that b is a subordinate of b, the sum $S(n)$ counts the total number of subordinate pairs (a, b) with $b < 10^n$, i.e., subordinate pairs (a, b) in which b (and hence a) has at most n decimal digits.

   Next, note that if a_i and b_i denote the i'th digits of a subordinate pair (a, b), respectively, then the number of possible choices for (a_i, b_i) is equal to

$$\#\{(h, k) : 0 \le h \le k \le 9\} = \sum_{k=0}^{9} (k + 1) = \frac{10 \cdot 11}{2} = 55.$$

   By padding integers with fewer than n digits on the left by 0's, we see that the subordinate pairs (a, b) with at most n decimal digits are in one-to-one correspondence with pairs of n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ of integers, with a_i, b_i satisfying $0 \le a_i \le b_i \le 9$. By the above observation, there are 55^n such pairs of tuples.

5. Let $a_1, a_2, \ldots, a_{65}$ be positive integers, none of which has a prime factor greater than 13. Prove that, for some i, j with $i \neq j$, the product $a_i a_j$ is a perfect square.

   Solution. There are 6 primes ≤ 13: $2, 3, 5, 7, 11, 13$. Denote these primes by $p_1, \ldots, p_6$. Then each of the given integers can be represented uniquely in the form $\prod_{i=1}^{6} p_i^{\alpha_i}$ with nonnegative integer exponents α_i. Now classify the given integers according to the parity of the vector of exponents $(\alpha_1, \ldots, \alpha_6)$. There are $2^6 = 64$ possible parity vectors. Since we have 65 integers, by the pigeonhole principle, two of these integers, say $a = \prod_{i=1}^{6} p_i^{\alpha_i}$ and $b = \prod_{i=1}^{6} p_i^{\beta_i}$, must have exponent vectors of the same parity, i.e., $\alpha_i \equiv \beta_i \bmod 2$ for all i. But then $\alpha_i + \beta_i \equiv 0 \bmod 2$, so the product $ab = \prod_{i=1}^{6} p_i^{\alpha_i+\beta_i}$ has only even exponents in its prime factorization and therefore must be a perfect square.

6. Let n be an *even* positive integer, and let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. Given two permutations $\sigma_1, \sigma_2 \in S_n$, define their distance $d(\sigma_1, \sigma_2)$ by

$$d(\sigma_1, \sigma_2) = \sum_{k=1}^{n} |\sigma_1(k) - \sigma_2(k)|$$

   Determine, with proof, the maximal distance between two permutations in S_n, i.e., determine the exact value of $\max_{\sigma_1, \sigma_2 \in S_n} d(\sigma_1, \sigma_2)$.

   Solution. We will show that the maximal distance is $n^2/2$ when n is even. (When n is odd, a similar argument gives $(n^2 - 1)/2$.)

   We first show that $n^2/2$ is an upper bound for the distance $d(\sigma_1, \sigma_2)$. To this end, fix two permutations σ_1 and σ_2. The sum defining $d(\sigma_1, \sigma_2)$ can be written as

$$d(\sigma_1, \sigma_2) = \sum_{k \in I} (\sigma_1(k) - \sigma_2(k)) - \sum_{k \in I^c} (\sigma_1(k) - \sigma_2(k)) \tag{1}$$

   where I is some (possibly empty) subset of the index set $\{1, 2, \ldots, n\}$, and I^c its complement. (In fact, I consists of those indices k for which $\sigma_1(k) - \sigma_2(k) \ge 0$, though we will not make use of this.) Using the notation

$$\Sigma_i(I) = \sum_{k \in I} \sigma_i(k),$$

   we can write (1) as

$$d(\sigma_1, \sigma_2) = \Sigma_1(I) - \Sigma_2(I) - \Sigma_1(I^c) + \Sigma_2(I^c). \tag{2}$$

   Now, note that

$$\Sigma_1(I^c) = \sum_{k=1}^{n} \sigma_1(k) - \Sigma_1(I) = \sum_{k=1}^{n} k - \Sigma_1(I) = \frac{n(n+1)}{2} - \Sigma_1(I), \tag{3}$$

and similarly

$$\Sigma_2(I) = \frac{n(n+1)}{2} - \Sigma_2(I^c). \tag{4}$$

Substituting (3) and (4) into (2) gives

$$\begin{aligned} d(\sigma_1, \sigma_2) &= 2(\Sigma_1(I) + \Sigma_2(I^c)) - n(n+1) \\ &= 2\left(\sum_{k\in I}\sigma_1(k) + \sum_{k\in I^c}\sigma_2(k)\right) - n(n+1). \end{aligned} \tag{5}$$

Now note that, since σ_1 and σ_2 are permutations of $\{1, 2, \dots, n\}$, the latter two sums contain exactly $|I| + |I^c| = n$ terms, all from the set $\{1, 2, \dots, n\}$. Moreover, any given element of $\{1, 2, \dots, n\}$ can occur at most twice (namely, as $\sigma_1(k)$ for some $k \in I$ and as $\sigma_2(k')$ for some $k' \in I^c$). It follows that the n terms in these two sums can include at most two n's, at most two terms $n - 1$, etc. Hence the sum of these terms is bounded by

$$\begin{aligned} \sum_{k\in I}\sigma_1(k) + \sum_{k\in I^c}\sigma_2(k) &\leq 2n + 2(n-1) + \cdots + 2(n/2+1) \\ &= 2\sum_{k=n/2+1}^{n} k = n(n+1) - (n/2)(n/2+1) \end{aligned} \tag{6}$$

Substituting (6) into (5) gives

$$d(\sigma_1, \sigma_2) \leq 2n(n+1) - n(n/2+1) - n(n+1) = \frac{n^2}{2},$$

as desired.

To show that the bound $n^2/2$ is attained, define σ_1 by $\sigma_1(k) = n/2 + k$, reducing modulo n if $n/2 + k > n$, and let σ_2 be the identity permutation, i.e., $\sigma_2(k) = k$ for all k. Then $|\sigma_1(k) - \sigma_2(k)| = n/2$ for each $k = 1, 2, \dots, n$, so

$$d(\sigma_1, \sigma_2) = \sum_{k=1}^{n}(n/2) = \frac{n^2}{2}.$$

Thus, $n^2/2$ is indeed the maximal distance between two permutations in S_n.