Putnam Practice Test 2

Oct. 26, 2009

1. Prove that there do *not* exist positive integers x, y, z, such that

$$x^2 + y^2 + z^2 = 2xyz.$$

2. Let $P(x)$ be a polynomial of degree 2009 satisfying $P(k) = k$ for $k = 1, \ldots, 2009$ and $P(0) = 1$. Find $P(-1)$.

3. [A3, Putnam 1985] Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$, by the condition

$$a_m(0) = \frac{d}{2^m}, \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{m\to\infty} a_m(m)$.

4. [B1, Putnam 1971] Let S be a set and $*$ a binary operation on S satisfying $x * x = x$ for all $x \in S$ and $(x * y) * z = (y * z) * x$ for all $x, y, z \in S$. Prove that $*$ is commutative (i.e., $x * y = y * x$ for all $x, y \in S$).

5. [A2, Putnam 1986] Determine the rightmost digit (in decimal) of $\left[\frac{10^{20000}}{10^{100}+3}\right]$.

6. [A2, Putnam 1984] Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

[Solutions at `http://www.math.uiuc.edu/contests.html`]