Putnam Practice Test 2

Oct. 26, 2009
Solutions

1. Prove that there do *not* exist positive integers x, y, z, such that

$$x^2 + y^2 + z^2 = 2xyz.$$

Solution. Let $t \geq 0$ be the highest power of 2 that divides each of the numbers x, y, and z, and let $X = x/2^t$, $Y = y/2^t$, and $Z = z/2^t$. The given equation, when divided by 2^{2t}, becomes

$$X^2 + Y^2 + Z^2 = 2^{t+1}XYZ,$$

and at least one of X, Y, Z is odd. The right hand side is an even number times XYZ. If exactly one or all three of the numbers X, Y, Z are odd, then the left side of the equation is odd while the right side is even, so these cases are impossible. In the remaining case exactly two of the numbers X, Y, Z are odd, and the other number is even. In this case, XYZ is even, so the right side of the equation is divisible by 4. On the other hand, since a square of an even number is conguent to 0 modulo 4, and a square of an odd number is conguent to 1 modulo 4, the left-hand side of the equation is congruent to 2 modulo 4. Hence it can not be divisible by 4, so we again arrive at a contradiction. Hence the given equation does not have a solution in integers.

2. Let $P(x)$ be a polynomial of degree 2009 satisfying $P(k) = k$ for $k = 1, \ldots, 2009$ and $P(0) = 1$. Find $P(-1)$.

Solution. We will show that $P(-1) = 2009$. More generally, we will show that if $P(x)$ is a polynomial of degree n with $P(k) = k$ for $k = 1, \ldots, n$ and $P(0) = 1$, then $P(-1) = n$. Let $Q(x) = P(x) - x$. Then $Q(x)$ is a polynomial of degree n whose roots are $k = 1, 2, \ldots, n$. Thus, $Q(x)$ is of the form $Q(x) = c\prod_{k=1}^n (x-k)$ for some constant c, and $P(x) = x + c\prod_{k=1}^n (x-k)$. Setting $x = 0$ gives $1 = P(0) = c(-1)^n n!$, so $c = (-1)^n/n!$. Hence

$$P(-1) = -1 + \frac{(-1)^n}{n!}\prod_{k=1}^n (-1-k) = -1 + (n+1) = n.$$

3. [A3, Putnam 1985] Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$, by the condition

$$a_m(0) = \frac{d}{2^m}, \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{m\to\infty} a_m(m)$.

Solution. Observe that the given recurrence relation can be written as

$$a_m(j+1)+1=(a_m(j)+1)^2, \quad j\geq 0.$$

Thus, setting $b_m(j)=a_m(j)+1$, we have

$$b_m(j)=b_m(j-1)^2=b_m(j-1)^{2^2}=\cdots=b_m(0)^{2^j}.$$

Together with the initial condition $b_m(0)=1+a_m(0)=1+d2^{-m}$ this gives

$$b_m(j)=\left(1+\frac{d}{2^m}\right)^{2^j},$$

for any j. Hence

$$\lim_{m\to\infty} b_m(m)=\lim_{m\to\infty}\left(1+\frac{d}{2^m}\right)^{2^m}=e^d,$$

and therefore

$$\lim_{m\to\infty} a_m(m)=\lim_{m\to\infty}(b_m(m)-1)=e^d-1.$$

4. [B1, Putnam 1971] Let S be a set and $*$ a binary operation on S satisfying $x*x=x$ for all $x\in S$ and $(x*y)*z=(y*z)*x$ for all $x,y,z\in S$. Prove that $*$ is commutative (i.e., $x*y=y*x$ for all $x,y\in S$).

   Solution. Applying the two given rules repeatedly, we have, for any $x,y\in S$,

$$\begin{aligned}x*y&=(x*y)*(x*y)=((x*y)*x)*y=((y*x)*x)*y=((x*x)*y)*y\\&=(x*y)*y=(y*y)*x=y*x,\end{aligned}$$

   which proves the commutativity of $*$.

5. [A2, Putnam 1986] Determine the rightmost digit (in decimal) of $\left[\frac{10^{20000}}{10^{100}+3}\right]$.

   Solution. Let x denote the number in brackets. Expanding $(1+3\cdot 10^{-100})^{-1}$ into a geometric series, we obtain

$$x=\sum_{n=0}^{\infty}(-1)^n3^n10^{19{,}900-100n}.$$

   In the last sum, all terms with $n<199$ are all divisible by 10 and the term $n=199$ equals $(-3)^{199}$. Also, since the series is alternating with decreasing terms, the sum of the terms with $n\geq 200$ is positive and bounded from above by the first of these terms, i.e., $3^{200}10^{-100}$, which is less than 1. Thus, the last digit of $[x]$ is equal to the last digit of $N=(-3)^{199}$ or, equivalently, the residue of N modulo 10. Since $(-3)^4\equiv 1$ modulo 10, we have $(-3)^{199}\equiv(-3)^3\equiv 3$ modulo 10, so the rightmost digit of $[x]$ is 3.

6. [A2, Putnam 1984] Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

Solution. Let S denote the given infinite series, and let a_k denote the kth term in this series. Then

$$a_k = \frac{(2/3)^k}{3(1 - (2/3)^{k+1})(1 - (2/3)^k)} = \frac{1}{1 - (2/3)^k} - \frac{1}{1 - (2/3)^{k+1}}.$$

Thus, the given series "telescopes," and its partial sums are given by

$$\sum_{k=1}^{n} a_k = \frac{1}{1 - (2/3)} - \frac{1}{1 - (2/3)^{n+1}}.$$

Letting $n \to \infty$, the last expression tends to 2. Thus, $S = 2$.