

# U OF I MOCK PUTNAM EXAM

SEPT. 29, 2009

1. Suppose  $P(x)$  is a polynomial with integer coefficients such that none of the values  $P(1), \dots, P(2009)$  is divisible by 2009. Prove that  $P(n) \neq 0$  for all integers  $n$ .
2. Find a function  $f(x)$  that satisfies, for all  $x \geq 0$ ,

$$f(x) = \sqrt{\int_0^x (f(t)^2 + f'(t)^2) dt} + 2009.$$

3. Let  $\mathcal{A} = (a_1, a_2, a_3, \dots)$  be a permutation of the positive integers. (In other words,  $a_k$  is a positive integer for each  $k$ , and for each positive integer  $n$ , there exists exactly one  $k$  such that  $a_k = n$ .) Prove that  $\mathcal{A}$  contains a triple  $(a_i, a_j, a_k)$  with  $i < j < k$  and  $a_j - a_i = a_k - a_j = d > 0$ .
4. Let  $b_1, \dots, b_n$  be integers greater than 1, and let  $B = b_1 \dots b_n$  denote their product. Let  $d_i$  be the number of digits of  $B$  expressed in base  $b_i$ . For example, if  $b_1 = 10$ ,  $b_2 = 3$ ,  $b_3 = 2$ , then  $B = 10 \cdot 3 \cdot 2 = (60)_{10} = (2020)_3 = (111100)_2$ , so  $d_1 = 2$ ,  $d_2 = 4$ ,  $d_3 = 6$ . Show that

$$d_1 + \dots + d_n > n^2.$$

5. Show that any positive integer containing exactly 2009 digits (in decimal), none of whose digits is zero, is either divisible by 2009, or can be changed to an integer that is divisible by 2009 by replacing some, but not all, of its digits by 0.
6. Let  $\mathcal{A}$  be an infinite set of positive integers, and let  $A(n)$  denote the number of elements of  $\mathcal{A}$  that are  $\leq n$ . Suppose that the series

$$\sum_{a \in \mathcal{A}} \frac{1}{a}$$

converges. Show that  $\lim_{n \rightarrow \infty} A(n)/n = 0$ .

[Solutions at <http://www.math.uiuc.edu/contests.html>]