

Mock Putnam Exam 1

October 21, 1996

Leo Moser had signed a contract with a publisher to write a book (a collection of problems). Every year the publisher's representative asked Leo, "When will you have the manuscript completed?" On the seventh such request, Leo replied, "I'll publish the book posthumously," to which the representative replied, "Well, okay, but make it soon!" (Moser died in 1970 at the age of 49; the book apparently was never published.)

1. Let a, b, c, d be positive integers for which $ab = cd$. Show that $a^2 + b^2 + c^2 + d^2$ is a composite number.
2. Prove that for any positive integers m and n , $m^{-1/n} + n^{-1/m} > 1$.
3. Let u be a real number with $0 < u < 1$. Let $u_0 = u$, and for $n \geq 1$ define u_n recursively by

$$u_n = \frac{1}{u_{n-1}} + u.$$

Prove that the sequence $\{u_n\}_{n \geq 1}$ converges and find its limit.

4. Find the sum of the series

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$$

5. A polynomial $P(x)$ of degree n satisfies $P(k) = 2^k$ for $k = 0, 1, 2, \dots, n$. Determine $P(n+1)$.
6. Prove that any positive integer n can be expressed in the form

$$n = \epsilon_1 1^2 + \epsilon_2 2^2 + \cdots + \epsilon_m m^2,$$

where m is a positive integer and $\epsilon_i = 1$ or $\epsilon_i = -1$ for $i = 1, 2, \dots, m$.