UIUC Department of Mathematics

Solutions to Mock Putnam Exam 1

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Problem 1. Find all integer solutions of the equation $a^2 + b^2 + c^2 = a^2b^2$.

Solution. Trivially, $(a, b, c) = (0, 0, 0)$ is a solution. We show that there are no other solutions. Suppose that (a, b, c) is an integer solution with at least one of a, b, and c non-zero. Let 2^k denote the highest power of 2 dividing each of a, b and c, and set $\bar{a} = a2^{-k}$, $\bar{b} = b2^{-k}$, and $\bar{c} = c2^{-k}$. Then, $(*)$ $\bar{a}^2 + \bar{b}^2 + \bar{c}^2 = 2^{2k}\bar{a}^2\bar{b}^2$, and at least one of $\bar{a}$, $\bar{b}$, and $\bar{c}$, is odd. Since a square can only be congruent to 0 or 1 modulo 4, it follows that the left-hand side of $(*)$ is congruent to 1, 2, or 3 modulo 4. On the other hand, the right-hand side is congruent to 0 modulo 4 unless both $\bar{a}$ and $\bar{b}$ are odd and $k = 0$. In the latter case, however, the right-hand side is congruent to 1 modulo 4, while the left-hand side is congruent to 2 or 3 modulo 4. Thus, $(*)$ has no integer solutions, and $(0, 0, 0)$ is the only solution to the original equation.

Problem 2. Show that the number $3^n + 4^n$ is divisible by 7 whenever n is an odd natural number.

Solution. This is easily proved using congruences. Starting from the congruence $3 \equiv -4$ modulo 7, we get $3^n \equiv (-4)^n$ modulo 7 for every positive integer n which means that $3^n - (-4)^n$ is divisible by 7. Since $3^n - (-4)^n = 3^n + 4^n$ for odd values of n, the claim follows.

Problem 3. Prove that if a pentagon (five-sided polygon) inscribed in a circle has equal angles, then its sides are equal.

Solution. This is based on the fact from elementary trigonometry that given any 3 points A, B and C on a circle, the length of arc(ABC) (the arc from A to C via B) is uniquely determined by the angle between the chords AB and BC. Denoting the vertices of the pentagon (in order) by A, B, C, D, and E, it follows from the given hypotheses that arc(ABC) and arc(BCD) have the same length. Thus, arc(AB) + arc(BC) = arc(BC) + arc(CD), which implies that arc(AB) = arc(CD). Similarly, we get arc(CD) = arc(EA) = arc(BC) = arc(DE). Thus the arcs joining consecutive vertices have the same lengths. This implies that the corresponding chords also have the same lengths, i.e., that the pentagon is equilateral.

Problem 4. Let $x_0 = 0$, $x_1 = 1$, and

$$x_{n+1} = \frac{x_n + nx_{n-1}}{n+1} \quad (n \geq 1).$$

Show that the sequence $\{x_n\}$ converges as $n \to \infty$ and determine its limit.

Solution. The given recurrence can be written as

$$x_{n+1}(n+1) = x_n + nx_{n-1} \quad (n \geq 1).$$

Setting $d_n = x_{n+1} - x_n$ and simplifying, we deduce $d_n = (-n/(n+1))d_{n-1}$ for $n \geq 1$. Iterating this relation, we get

$$d_n = \frac{-n}{n+1} \cdot \frac{-(n-1)}{n} \cdots \frac{-1}{2} d_0 = \frac{(-1)^n}{n+1},$$

since $d_0 = x_1 - x_0 = 1$. Hence

$$x_n = x_0 + \sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} \frac{(-1)^i}{i+1}.$$

The last series is an alternating series with decreasing terms and thus converges. Its sum, $\sum_0^\infty (-1)^i/(i+1)$ equals $\ln 2$ by the Taylor series for $\ln(1+x)$.

Problem 5. Evaluate the integral $\int_0^1 \ln x \ln(1-x) dx$. (You may use the formula $\sum_{n=1}^\infty n^{-2} = \pi^2/6$.)

Solution. Let I denote the integral to be evaluated. Substituting the Taylor series $\ln(1-x) = -\sum_{n=1}^\infty x^n/n$ into the integral, we get **formally** *

$$I = -\sum_{n=1}^\infty \frac{1}{n} \int_0^1 x^n \ln x dx.$$

An integration by parts shows that the last integral equals $-(n+1)^{-2}$. Hence

$$I = \sum_{n=1}^\infty \frac{1}{n(n+1)^2} = \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right).$$

In the last sum, the first two terms yield a telescoping series with sum 1, while the contribution of the third term is $-\sum_{n=1}^\infty (n+1)^{-2} = -(\pi^2/6 - 1)$ by the formula given in the problem. Hence $I = 2 - \pi^2/6$.

* This argument is not completely rigorous. For a rigorous proof, one would have to (i) restrict the range of integration to $(\epsilon, 1-\epsilon)$ (and let $\epsilon \to 0$ at the end of the argument), and (ii) use the finite Taylor expansion $\ln(1-x) = -\sum_{n=1}^k x^k/n + R_k(x)$, along with a bound for the remainder $R_k(x)$, and let $k \to \infty$ **after** interchanging summation and integration.