2010 U OF I MOCK PUTNAM EXAM

Solutions

1. (a) Given a set with n elements (where n is a positive integer), prove that exactly 2^{n-1} of its subsets have an odd number of elements.

   (b) Determine, with proof, the number of 8 by 8 matrices in which each entry is 0 or 1 and each row and each column contains an odd number of 1's.

   Solution. (a) Let N_0 and N_1 denote the number of subsets of an n-element set with an even, respectively odd, number of elements. Clearly $N_0 + N_1 = 2^n$ and

$$N_0 = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k}, \quad N_1 = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k}.$$

   Moreover, by the binomial theorem,

$$N_0 - N_1 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0,$$

   so $N_0 = N_1$. Since $N_0 + N_1 = 2^n$, it follows that $N_0 = N_1 = 2^{n-1}$.

   (b) The number of ways to fill a given row of 8 entries with 0's and 1's using an odd number of 1's is equal to the number of subsets of an 8-element set with an odd number of elements (as can be seen by labeling the entries with $1, 2, \ldots, 8$ and identifying a given row with the set of labels from $\{1, 2, \ldots, 8\}$ for which the corresponding entries are equal to 1). By part (a), this number is 2^7. The number of ways to fill the first 7 rows of the 8×8 matrix in this way is $(2^7)^7 = 2^{49}$. Each of these 2^{49} 7×8 matrices then has an odd number of 1's in each row, and there is exactly one way to add an 8th row to this matrix so that the number of 1's in each column is odd. Furthermore, if we choose the 8th row in this way, then the total number of 1's in the matrix is even (since it is a sum of eight odd numbers), and since the number of 1's in the first 7 rows is odd (as a sum of 7 odd numbers), it follows that the number of 1's in the last row must be odd as well. Thus, all rows and all columns in this matrix have an odd number of 1's, and therefore the matrices with the requested properties are exactly the 2^{49} matrices constructed above.

2. A sheet of paper contains the numbers $101, 102, \ldots, 200$. Suppose you play the following game on this list of numbers. At each stage, you pick two of the numbers on the list, say a and b, cross these out, and replace them by the single number $ab + a + b$. You keep doing this until only a single number is left (which happens after 99 such moves). Determine, with proof, what this last number is.

   Solution. The answer is $201!/101! - 1$. For the proof, the key observation is that if a pair of numbers (a, b) on the list is replaced by $ab + a + b$, the product of all the numbers on the list, each incremented by 1, remains unchanged since $(a+1)(b+1) = ((ab + a + b) + 1)$. It follows that, if N denotes the last surviving number, then $N + 1$ is equal to the product of the original 100 numbers, each incremented by 1, i.e., $N + 1 = \prod_{k=101}^{200}(k+1) = 201!/101!$. Hence $N = 201!/101! - 1$.

3. Among all powers of 2, what percentage begin with the digit 1 in their decimal representation? More precisely, if $f(n)$ denotes the number of integers among the first n powers of 2 (i.e., $2^1, 2^2, \ldots, 2^n$) whose decimal representation begins with the digit 1, show that the limit $\lim_{n\to\infty} f(n)/n$ exists and compute its value.

   Solution. The limit in question is $\log_{10} 2$. For the proof, observe that (i) any interval of the form $[x, 2x)$ with $x \geq 2$ contains exactly one power of 2, and (ii) a positive integer begins with the digit 1 in its decimal representation if and only if it falls into an interval of the type $[10^k, 2 \cdot 10^k)$, for some nonnegative integer k. These observations imply that each of the intervals $[10^{k-1}, 10^k)$, $k = 1, 2, \ldots$, contains exactly one power of 2. Hence, $f(n)$, the number of powers of 2 that are $\leq 2^n$, is equal to the smallest integer k satisfying $10^k > 2^n$. Rewriting the latter inequality as $k > n \log_{10} 2$, we see that k is the ceiling of $n \log_{10} 2$, so

$$f(n) = \lceil n \log_{10} 2 \rceil.$$

Therefore the limit in question is

$$\lim_{n\to\infty} \frac{f(n)}{n} = \lim_{n\to\infty} \frac{\lceil n \log_{10} 2 \rceil}{n} = \log_{10} 2.$$

Remarks: (i) Using more advanced methods (namely, the theory of "uniform distribution modulo 1"), one can show that given any positive integer d, the proportion of powers of 2 whose decimal representation begins with the digits of d is $\log_{10}(d+1)/d$. Thus, for instance, a proportion of $\log_{10}(8/7) = 0.0579919\ldots$ or about 5.78 percent, of all powers of 2 begin with the digit 7, and proportion of $\log_{10}(2011/2010) = 0.000216\ldots$, or about 0.0216 percent, of powers of 2 begin with the digits 2010. A remarkable consequence of this result is that, given any finite sequence of digits (for example, 999999 or 31415926) there exists a power of 2 that begins that sequence.

(ii) Computing the first few terms of the sequence 2^n one might suspect that every third term begins with a 1, but this pattern does not persist, and in fact, there is no periodicity in the sequence of leading digits. (If there were, the limit $\lim_{n\to\infty} f(n)/n$ would be a rational number, but its true value, $\log_{10} 2 = 0.30103\ldots$ is irrational.)

4. Given a positive integer d, define a *lattice traversal of step size* d to be an infinite polygonal path $P_0P_1P_2\ldots$ in the plane satisfying the following conditions:

   (i) The distance between any two consecutive points P_i and P_{i+1} on the path is d.

   (ii) Each point P_i on the path is a lattice point (i.e., has integer coordinates).

   (iii) Each lattice point in the plane occurs at least once as a point P_i on the path.

   Determine, with proof, for which integers $d \in \{2, 3, \ldots, 10\}$ there exists a lattice traversal of step size d.

   Solution. We show that $d = 5$ is the only number in the given set with this property. First note that if $d \neq 5$ and $d \neq 10$, then d is not the hypothenuse of a right triangle with integer sides, so the only moves of length d between lattice points are moves in horizontal or vertical direction. These moves leave the remainder of the x and y-coordinates modulo d unchanged, so all points P_i on the path must have x and y-coordinates congruent to the x and y-coordinates of the starting point P_0. Since $d \geq 2$, these points do not represent all lattice points, so the path is not a lattice traversal in the sense of the problem.

   If $d = 10$, then the possible moves are horizontal or vertical moves of length 10 and diagonal moves in direction $(\pm 6, \pm 8)$ or $(\pm 8, \pm 6)$. In either case the parity of the coordinates remains the same, so lattice points with parity different from that of the starting point cannot be reached.

   Next, we will show that a lattice traversal of step size $d = 5$ does exist. We take P_0 to be the origin and first observe that it is enough to show that we can get from the origin to the point $(1, 0)$ in finitely many allowable moves. For suppose we have a path of the desired type from $(0, 0)$ to $(1, 0)$. By rotating this path by 90, 180, or 270 degrees, we can then also reach the points $(0, 1)$, $(-1, 0)$, and $(0, -1)$ from the origin. By repeated applications of such moves, shows that we can reach any lattice point from any other lattice point in finitely many moves. To obtain a lattice traversal, enumerate all lattice points in some sequence (this is possible since the set of lattice points has cardinality $\mathbb{Z} \times \mathbb{Z}$ and is therefore countable), and connect any two adjacent lattice points in this sequence by such a finite sequence of moves.

   Thus, it remains to construct a sequence of moves of step size 5 from $(0, 0)$ to $(1, 0)$. The sequence given by $(0, 0) \to (3, 4) \to (-2, 4) \to (1, 0)$ does the job. Therefore, the proof is complete.

5. Let $1 \leq a_1 < a_2 < a_3 \ldots$ be a sequence of positive integers, such that $a_k/k \to \infty$ as $k \to \infty$, and let $A(n)$ denote the number of terms in this sequence that are $\leq n$. Prove that there exist infinitely many positive integers n that are divisible by $A(n)$.

   Solution. We begin by showing that $A(n)/n \to 0$ as $n \to \infty$. Indeed, given $n \geq a_1$, let k be defined by $a_k \leq n < a_{k+1}$. Then $A(n) = k$, and

   (1) $$\frac{A(n)}{n} = \frac{k}{n} \leq \frac{k}{a_k} \to 0,$$

   as $n \to \infty$, by the hypothesis $a_k/k \to \infty$.

   Next, we will show that, for every integer $d \geq a_1$, there exists $m = m_d$ such that

   (2) $$m = A(dm).$$

   Since the numbers $n = md$ for which (2) holds have the property that $A(n)$ divides n, this will prove the desired result.

   To obtain (2), fix $d \geq a_1$ and consider the function $f(m) = f_d(m) = A(dm)/m$. Note that $f(1) = A(d) \geq A(a_1) \geq 1$ and, by (1), $f(m) = A(dm)/m \to 0$ as $m \to \infty$. Hence, there exists a maximal integer $m = m_d$ such that $f(m) \geq 1$. We claim that (2) holds with this choice of m. To see this, note that, by the maximality of m and the monotonicity of the function $A(n)$, we have $f(m+1) < 1$ and hence $A(dm) \le A(d(m+1)) < m+1$, so $A(dm) \le m$, or $f(m) = A(dm)/m \le 1$. Combined with our earlier inequality $f(m) \ge 1$ this implies $f(m) = 1$, which is equivalent to (2).

6. Find, with proof, the precise set of real numbers α, such that any sequence x_n, $n = 1, 2, 3, \ldots$, of real numbers satisfying

(1) $$\lim_{n\to\infty} (x_n - x_{n-2}) = 0.$$

also satisfies

(2) $$\lim_{n\to\infty} \frac{x_n}{n^\alpha} = 0.$$

Solution. The answer is $\alpha \ge 1$. Clearly, if the statement holds for a particular value of α, then it also holds for any larger value of α. Thus, it suffices to (i) prove that any sequence x_n satisfying (1) satisfies (2) with $\alpha = 1$, and (ii) find an example of a sequence $\{x_n\}$ satisfying (1), which does not satisfy (2) for any $\alpha < 1$.

(i) Proof that (1) implies (2) with $\alpha = 1$: (This appeared on the Putnam (Problem A4, 1970).)

Let $\{x_n\}$ be a given sequence satisfying (1), and write $d_n = x_n - x_{n-2}$. Let $\epsilon > 0$ be given. By the assumption (1), there exists $n_0 = n_0(\epsilon) \ge 3$ such that $|d_n| < \epsilon$ for all $n \ge n_0$. For $n \ge n_0$ we have

$$\begin{aligned} |x_{2n+1}| &= \left| \sum_{i=1}^{n} d_{2i+1} + x_1 \right|, \\ &\le \sum_{k=n_0}^{2n+1} |d_k| + \sum_{i=3}^{n_0} |d_i| + |x_1|, \\ &\le \epsilon(2n+1) + C(n_0), \end{aligned}$$

say, where $C(n_0)$ is a constant depending on n_0 (and hence on ϵ), but not on n. Dividing both sides by $2n+1$ and letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{|x_{2n+1}|}{2n+1} \le \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that $\lim_{n\to\infty} |x_{2n+1}|/(2n+1) = 0$. A completely analogous argument gives $\lim_{n\to\infty} |x_{2n}|/(2n) = 0$, and combining these two relations shows that (2) holds with $\alpha = 1$.

(ii) Counterexample for $\alpha < 1$: Let $x_1 = 1$, and for $n \ge 2$, define x_n by $x_n = n/\log n$. Then $x_n/n^\alpha = n^{1-\alpha}/\log n$, which is unbounded when $\alpha < 1$, so (2) fails for such α. On the other hand, the sequence satisfies (1), since

$$\begin{aligned} |x_n - x_{n-2}| &= \left| \frac{n}{\log n} - \frac{n-2}{\log(n-2)} \right| \\ &\le 2 \max_{n-2 \le x \le n} \left| \left(\frac{x}{\log x} \right)' \right| \le \frac{2}{\log(n-2)}, \end{aligned}$$

which tends to 0 as $n \to \infty$.