

# 2011 U OF I MOCK PUTNAM CONTEST

1. [Variation of A2, Putnam 1987] The sequence of digits

1 2 3 4 5 6 7 8 9 1 0 1 1 1 2 1 3 ...

is obtained by writing out the natural numbers in order. Let  $f(n)$  denote the position of the first digit of the number  $n$  in this sequence. Thus, for example,  $f(1) = 1$ ,  $f(2) = 2$ ,  $f(10) = 10$  (since the integer  $n = 10$  occupies positions 10 and 11 in this sequence),  $f(11) = 12$  (since 11 occupies positions 12 and 13),  $f(12) = 14$  (since 12 occupies positions 14 and 15, and so on.

Find, with proof, a simple explicit formula for  $f(10^k)$ , where  $k$  is an arbitrary positive integer.

2. Let  $a_n = [(\sqrt{2} + 1)^n]$ , where  $[x]$  denotes the greatest integer  $\leq x$  (i.e., the floor function). Prove that  $a_n$  is even if and only if  $n$  is odd.
3. [A2, Putnam 2003] Let  $a_1, \dots, a_n, b_1, \dots, b_n$  be positive real numbers. Prove that

$$(a_1 \dots a_n)^{1/n} + (b_1 \dots b_n)^{1/n} \leq (a_1 + b_1)^{1/n} \dots (a_n + b_n)^{1/n}.$$

4. Let  $P_1, P_2, P_3, \dots$  be a sequence of points in 3-dimensional space satisfying (i)  $|P_i| \geq 1$  for all  $i$  and (ii)  $|P_i P_j| \geq 1$  for all  $i$  and  $j$  with  $i \neq j$ . (Here  $|PQ|$  denotes the usual (Euclidean) distance between  $P$  and  $Q$ , and  $|P|$  denotes the distance between  $P$  and the origin.)

(a) Prove that, if  $\alpha > 3$ , then the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{|P_i|^\alpha}$$

converges.

(b) Show that there exists a sequence of points  $P_i$  satisfying conditions (i) and (ii) above for which the above series diverges when  $\alpha = 3$ .

5. Determine, with proof, whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{2+\cos(2\pi \ln n)}}$$

converges.

6. Let  $G_n$  denote the geometric mean of the  $n$  binomial coefficients  $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$ .

Prove that the limit  $\lim_{n \rightarrow \infty} \sqrt[n]{G_n}$  exists, and find its value. (You may only use the definition of binomial coefficients and standard results from calculus, but not, for example, asymptotic formulas for binomial coefficients or factorials.)

[Solutions at <http://www.math.uiuc.edu/contests.html>]