Mock Putnam Exam 1/2000

Elementary Version

E1. Show that for $n = 1, 2, 3, \ldots$

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^k = \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} 2^k + (-1)^n.$$

E2. Prove that there do **not** exist positive integers x, y, z, such that

$$x^2 + y^2 + z^2 = 2xyz.$$

E3. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers and let a new sequence $q_1, q_2, q_3, \ldots$ be defined by $q_1 = a_1$, $q_2 = a_2 q_1 + 1$, and $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 3$. Show that no two consecutive q_n's are even.

E4. A car travels from one city to another at the rate of 40 miles per hour and then returns at the rate of 60 miles per hour. What is the average rate for the round trip? Justify your answer.

E5. Find a general formula for the sum

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}.$$

Advanced Version

A1. Suppose that 15 distinct points $P_1, P_2, \ldots, P_{15}$ are chosen on the circumference of a circle, and all chords $P_i P_j$ with $1 \leq i < j \leq 15$ are drawn. It is easily seen that there are 105 such chords. Find the number of points *inside* the circle at which these chords intersect, assuming that no three chords intersect at the same point inside the circle.

A2. Find, with proof, the smallest value of the expression

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2,$$

among all pairs of positive real numbers (a, b) with $a + b = 1$.

A3. Determine, with proof, the set of all positive real numbers a for which the inequality $a^x \geq x^a$ is true for all positive real numbers x.

A4. Show that the sequence

$$\sqrt{2},\ \sqrt{2 + \sqrt{2}},\ \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

converges and find its limit.

A5. Prove that the binomial coefficients $\binom{2^n - 1}{k}$, where $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^n - 1$, are all odd.