UIUC Mock Putnam Exam 1/2001

October 2, 2001

Elementary Problems

E1 Evaluate $f(n) = 1^2 - 2^2 + 3^2 - \cdots + (2n-1)^2 - (2n)^2$.

E2 Show that, if n is odd, then $1^n + 2^n + \cdots + n^n$ is divisible by n^2.

E3 Let $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, and for $n > 3$ define a_n by $a_n = a_{n-1}a_{n-3}$. Find a_{2001}.

E4 Evaluate the sum $\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k$.

Advanced Problems

A1 Let n and m be positive integers with $n \geq 2m$. How many binary strings of length n are there that contain exactly m blocks of the form 01?

A2 Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $\lim_{n\to\infty}(H_n - \log n) = 1 - \int_0^1 \left\{\frac{1}{x}\right\} dx$, where $\{y\}$ denotes the fractional part of y.

A3 Let $a_1 = \sqrt{2}$, and for $n > 1$ define a_n by $a_n = (\sqrt{2})^{a_{n-1}}$. Prove that the sequence $\{a_n\}$ converges and determine its limit.

A4 Find all polynomials $P(x)$, all of whose roots are real and which satisfy $(*)$ $P(x^2-1) = P(x)P(-x)$ for all x.