Mock Putnam Exam 1/2000
Solutions

Elementary Problems

Problem E1. Show that for $n = 1, 2, 3, \ldots$

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^k = \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} 2^k + (-1)^n.$$

Solution. By the binomial theorem,

$$\sum_{1}^{n} \binom{n}{k} x^k = (1+x)^n.$$

The choice $x = -2$ yields

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^k - \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} 2^k = (1-2)^n = (-1)^n.$$

Problem E2. Prove that there do **not** exist positive integers x, y, z, such that

$$x^2 + y^2 + z^2 = 2xyz.$$

Solution. Let $t \geq 0$ be the highest power of 2 that divides it each of the numbers x, y, and z, and let $X = x/2^t$, $Y = y/2^t$, and $Z = z/2^t$. The given equation, when divided by 2^{2t} becomes

$$X^2 + Y^2 + Z^2 = 2^{t+1} XYZ,$$

with at least one of X, Y, Z is odd. The right hand side is an even number times XYZ. If exactly one or if all three of the numbers X, Y, and Z are odd, then the left side of the equation is odd and right right side even, an impossibility. If exactly two of the numbers X, Y, and Z are odd and one even, then the right side of the equation is an exact multiple of 4, while the left side leaves a remainder of two upon division by 4, again an impossibility. In no case does the equation have a solution in integers.

Problem E3. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integ ers and let a new sequence $q_1, q_2, q_3, \ldots$ be defined by $q_1 = a_1$, $q_2 = a_2 q_1 + 1$, and $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 3$. Show that no two consecutive q_n's are even.

Solution. From the definition, if q_1 is even, then $q_2 = a_2 q_1 + 1$ is odd; thus it is not possible that q_1 and q_2 both be even. Now let q_I and q_{I-1} denote the *first* occurrence of consecutive even elements of the sequence $\{q_i\}$. By the previous remark, $I \geq 3$. From the definition of the sequence,

$$q_I - a_I q_{I-1} = q_{I-2} \text{ also is even,}$$

violating of the minimality of I; thus there do not exist consecutive even members of the sequence.

Problem E4. A car travels from one city to another at the rate of 40 miles per hour and then returns at the rate of 60 miles per hour. What is the average rat e for the round trip? Justify your answer.

Solution. Let D denote the distance between the cities (in miles). Then the total time for the trip is $T = D/40 + D/60$ hours. The average rate of speed for the round trip is

$$\frac{2D}{D/40 + D/60} = 48 \text{ mph.}$$

Problem E5. Find a general formula for the sum

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}.$$

Solution. The general formula is

$$S_n := \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} \;=\; 1 - \frac{1}{(n+1)!}.$$

The proof is by induction on n. For $n = 1$, we have $S_1 = 1/2 = 1 - 1/2!$. Assuming the validity of the formula for S_n,

$$S_{n+1} = S_n + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}.$$

Thus the formula holds for the $n + 1^{st}$ case, completing the induction.

Advanced Problems

Problem A1. Suppose that 15 distinct points $P_1, P_2, \ldots, P_{15}$ are chose n on the circumference of a circle, and all chords P_iP_j with $1 \leq i < j \leq 15$ are drawn. It is easily seen that there are 105 such chords. Fi nd the number of points *inside* the circle at which these chords intersect, assuming that no three chords intersect at the same point inside the circle.

Solution. Each four distinct points on the dircle determine a unique point of intersection of two lines lying inside the circle. Thus there are

$$\binom{15}{4} = 1365$$

points of intersection.

Problem A2. Find, with proof, the smallest value of the expression

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2,$$

among all pairs of positive real numbers (a, b) with $a + b = 1$.

Solution. By the Cauchy inequality, $x^2 + y^2 \geq (1/2)(x + y)^2$. Thus

$$S := \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{1}{2}\left(a + \frac{1}{a} + b + \frac{1}{b}\right)^2 = \frac{1}{2}\left(1 + \frac{1}{ab}\right)^2.$$

(Here we have used the condition that $a + b = 1$.) By the arithmetic-geometric mean inequality

$$\frac{1}{2} = \frac{a+b}{2} \geq \sqrt{ab}.$$

Thus $1/(ab) \geq 4$ and $S \geq (1/2)(1 + 4)^2 = 25/2$. Choosing $a = b = 1/2$ shows that $S = 25/2$ can be achieved.

Problem A3. Determine, with proof, the set of all positive real numbers a for which the inequality $a^x \geq x^a$ is true for all positive real numbers x.

Solution. We show that $a^{1/a} \geq x^{1/x}$ holds for a suitable $a > 0$ and all positive x by finding

$$\max_{a>0} a^{1/a}.$$

To do this, it suffices to maximize $f(a) := (1/a)\ln a$. We have $f'(a) = (1-\ln a)/a^2$, a function which is positive for $0 < a < e$ and negative for $a > e$. Thus the maximum of f, and hence the maximum of the original function, occurs at $a = e$. It follows that $e^{1/e} \geq x^{1/x}$ holds for all positive x.

Problem A4. Show that the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find its limit.

Solution. Let

$$a_n = \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}},$$

with n 2's. By induction, each member of the sequence $\{a_n\}$ is less than 2: $a_1 = \sqrt{2} < 2$. Now suppose $a_n < 2$. We have $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$. Thus the upper bound for $\{a_n\}$ is established. The inequality

$$2 + a_n - a_n^2 = (2-a_n)(1+a_n) > 0$$

implies that $2 + a_n > a_n^2$, and thus $a_{n+1} > a_n$. A monotone increasing sequence that is bounded from above has a limit; let L denote the limit of the sequence $\{a_n\}$. Letting $n \to \infty$ in the equation $a_{n+1} = \sqrt{2+a_n}$, we see that $L = \sqrt{2+L}$, whence $L = 2$.

Problem A5. Prove that the binomial coefficients $\binom{2^n-1}{k}$, where $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^n - 1$, are all odd.

Solution. We have

$$\binom{2^n-1}{k} = \frac{2^n-1}{1} \cdot \frac{2^n-2}{2} \cdot \frac{2^n-3}{3} \cdots \frac{2^n-(k-1)}{k-1} \cdot \frac{2^n-k}{k}.$$

The general factor of this equation is $(2^n - j)/j,\ 1 \leq j \leq k$. If we write $j = 2^s r$, with r an odd number and (obviously) $2^s \leq j < 2^n$, then we have

$$\frac{2^n - j}{j} = \frac{2^{n-s} - r}{r}.$$

We see that after reduction, each factor in the denominator is an odd number (obviously) and each in the numerator also odd, since $n - s \geq 1$. Since binomial coefficients are known to be integers, $\binom{2^n-1}{k}$ is an odd integer.