Mock Putnam Exam 2

November 11, 1996

Setting the questions is not a simple matter. They must not be too easy but also not so difficult as to induce a feeling of frustration. It is sometimes rather hard to decide whether a question is easy. It may appear simple enough when the solution is known. (L.J. Mordell, "The Putnam Competition," Amer. Math. Monthly 70 (1963), 483)

1. Let m and r be integers with $0 \leq r \leq m$. Find a simple formula for the sum

$$S = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^r \binom{n}{r}.$$

2. Let a be a positive real number. Find the length of the shortest chord that is normal to the parabola $y^2 = 2ax$ at one end.

3. Determine all polynomials $P(x)$ such that $P(x^2+1) = (P(x))^2 + 1$ and $P(0) = 0$.

4. Let $a_1, a_2, \ldots, a_{2n+1}$ be a set of $2n+1$ integers such that, if any one of them is removed, the remaining ones can be divided into two sets with equal sums. Prove that $a_1 = a_2 = \cdots = a_{2n+1}$.

5. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that there exist only finitely many real numbers $x > 0$ for which

$$\int_0^x P(t) \sin t dt = 0, \quad \int_0^x P(t) \cos t dt = 0.$$

6. Evaluate the infinite series

$$\sum_{n=1}^{\infty} \sin \frac{2x}{3^n} \sin \frac{x}{3^n}.$$