UIUC Department of Mathematics

Solutions to Mock Putnam Exam 2

October 13, 1997

Problem 1. Determine which of the two expressions $\sqrt{n}^{\sqrt{n+1}}$ and $\sqrt{n+1}^{\sqrt{n}}$ is bigger when n is an integer greater than 8.

Solution. Let $f(n)$ and $g(n)$ denote the first and second expressions, respectively, and let $d(n) = f(n)/g(n)$. We show that $\ln d(n)$ is positive for $n \geq 9$, which proves that $f(n)$ is greater than $g(n)$ in that range. A simple calculation gives

$$\ln d(n) = \frac{1}{2}\left(\left(\sqrt{n+1} - \sqrt{n}\right)\ln n - \sqrt{n}\ln\left(1 + \frac{1}{n}\right)\right).$$

Since $\sqrt{n+1} - \sqrt{n} = \int_n^{n+1}(2\sqrt{x})^{-1}dx > (2\sqrt{n+1})^{-1}$ and $\ln(1 + 1/n) < 1/n$ for all n, it follows that $\ln d(n) \geq \frac{\ln n}{2\sqrt{n+1}} - \sqrt{n}$. The latter expression is positive if and only if $(*)$ $\ln n > 2\sqrt{1 + 1/n}$. Now, for $n \geq 9$, we have $2\sqrt{1+1/n} < 2\sqrt{10/9}$ and $\ln n > 2\ln 3$, and since $(**)$ $\ln 3 > \sqrt{10/9}$, $(*)$ holds for such n. (The inequality $(**)$ can easily be checked with a calculator. Without using a calculator, the proof of $(**)$ takes more work.)

An alternative approach that doesn't require any numerical calculations is to rewrite the inequality $\ln f(n) > \ln g(n)$ in the equivalent form $\ln n/\sqrt{n} > \ln(n+1)/\sqrt{n+1}$ and establish the latter inequality for $n \geq 9$ by observing that the function $h(x) = x^{-1/2}\ln x$ is decreasing when $x > e^2$: We have $h'(x) = x^{-3/2}(1 - (1/2)\ln x)$ which is negative for $x > e^2$.

Problem 2. The Fibonacci numbers F_n are defined by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Evaluate $\sum_{n=1}^{\infty}(F_n F_{n+2})^{-1}$.

Solution. The trick here is to transform the series into one that telescopes. Using the identity

$$\frac{1}{F_n F_{n+2}} = \frac{1}{F_{n+2} - F_n}\left(\frac{1}{F_n} - \frac{1}{F_{n+2}}\right) = \frac{1}{F_{n+1}F_n} - \frac{1}{F_{n+2}F_{n+1}},$$

we get, for each positive integer N,

$$\sum_{n=1}^{N}\frac{1}{F_n F_{n+2}} = \sum_{n=1}^{N}\left(\frac{1}{F_{n+1}F_n} - \frac{1}{F_{n+2}F_{n+1}}\right) = \frac{1}{F_2 F_1} - \frac{1}{F_{N+2}F_{N+1}}.$$

The last term goes to zero when $N \to \infty$. Hence, the given infinite series converges and has sum $(F_2 F_1)^{-1} = 1$.

Problem 3. Evaluate $\int_0^{\pi/2} \log(\sin x)dx$.

Solution. Let I denote the integral of the problem. Substituting $x = \pi/2 - y$ and using the identity $\sin(\pi/2 - y) = \cos y$, we see that I is equal to the integral $\int_0^{\pi/2} \log \cos y dy$. Thus,

$$\begin{aligned} 2I &= \int_0^{\pi/2} (\log \sin x + \log \cos x)dx = \int_0^{\pi/2} \log(\sin x \cos x)dx \\ &= \int_0^{\pi/2} \log\left(\frac{1}{2}\sin(2x)\right)dx = \left(\log\frac{1}{2}\right)\frac{\pi}{2} + \int_0^{\pi/2} \log\sin(2x)dx. \end{aligned}$$

The change of variables $y = 2x$ shows that the last integral equals I. Solving the resulting equation for I yields $I = -(\pi/2)\log 2$.

Problem 4. Prove that the expression

$$\sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}}$$

converges (in an appropriate sense) and find its value.

Solution. Let a_n denote the above expression, truncated after the nth occurrence of 3. Thus, $a_1 = \sqrt{3}$ and $(*)$ $a_n = \sqrt{3 + a_{n-1}}$ for $n \geq 2$. We show that the sequence $\{a_n\}$ converges to the number $\alpha = (1 + \sqrt{13})/2$.

First note that $(*)$ implies that $a_n > \sqrt{3}$ for all $n \geq 1$. Now let $a_0 = 0$ and set $d_n = a_n - a_{n-1}$ for $n \geq 2$. By $(*)$ we have, for $n \geq 2$,

$$d_n(a_n + a_{n-1}) = a_n^2 - a_{n-1}^2 = a_{n-1} - a_{n-2} = d_{n-1},$$

and since $a_n + a_{n-1} \geq 2\sqrt{3} > 3$, this implies $d_n < d_{n-1}/3$ for all $n \geq 2$. Iterating this inequality, we see that for $n \geq 2$, $d_n < 3^{-n+1}d_1 = 3^{-n+3/2}$. It follows that the series $\sum_{k=1}^{\infty} d_k$ converges, and since $a_n = \sum_{k=1}^{n} d_k$, this proves the convergence of the sequence $\{a_n\}$.

Let α denote the limit of this sequence. Letting $n \to \infty$ on each side of $(*)$ we deduce $\alpha = \sqrt{3 + \alpha}$. Thus, α is the positive root of the quadratic equation $\alpha^2 = 3 + \alpha$, namely $\alpha = (1 + \sqrt{13})/2$.

Problem 5. Show that for every positive integer n the binomial coefficient $\binom{2n}{n}$ is divisible by $n + 1$.

Solution. Here is a one line proof:

$$\binom{2n}{n} = (2n+2)\binom{2n}{n} - (2n+1)\binom{2n}{n} = (2n+2)\binom{2n}{n} - (n+1)\binom{2n+1}{n},$$

and since each term on the right is divisible by $n + 1$, so must be the expression on the left.