UIUC Department of Mathematics

Mock Putnam Exam 2

October 26, 1998

This exam is intended as a practice test for the real Putnam Exam and will be graded in the same way. To receive credit, you need to explain yourself clearly and succinctly; an answer alone won't do.

Graded exams will be returned at next Monday's Putnam Training Session.

Solutions will be posted by the end of this week at http://www.math.uiuc.edu/~hildebr/putnam/mockputnam.html.

****

1. [AIME 1985] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$. Given that $a_{100} = 1$ and $a_{200} = 0$, what is a_{1998}?

2. (a) How many subsets of a set with 8 elements have an odd number of elements?
(b) [MIT] How many 8 by 8 matrices are there in which each entry is 0 or 1 and each row and each column contains an odd number of 1's?

3. [AIME 1988] For any positive integer k let $f_1(k)$ denote the sum of the squares of the digits of k (in decimal), and for $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1998}(11)$.

4. Let P(x) be a polynomial of degree n satisfying $P(k) = k$ for $k = 1, \ldots, n$ and $P(0) = 1$. Find $P(-1)$.

5. [MIT] Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers tending to 0 as $n \to \infty$, and let $b_n = a_n - 2a_{n+1} + a_{n+2}$. Assume that $b_n \geq 0$ for all n. Evaluate $\sum_{n=1}^{\infty} nb_n$.

Sources:

[MIT] MIT Problem Solving Seminar:
http://www-math.mit.edu/~ rstan/classes/S34/problems.html.

[AIME] American Invitational Mathematics Examination