UIUC Department of Mathematics

Mock Putnam Exam 2

October 26, 1998

Solutions

1. [AIME 1985] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$. Given that $a_{100} = 1$ and $a_{200} = 0$, what is a_{1998}?

Solution: Working out the first few terms of the sequence we get $a_3 = a_2 - a_1$, $a_4 = a_3 - a_2 = (a_2 - a_1) - a_2 = -a_1$, $a_5 = a_4 - a_3 = -a_2$, $a_6 = a_5 - a_4 = a_1 - a_2$, $a_7 = a_6 - a_5 = a_1$, and the pattern repeats itself from then on. Thus, the sequence is periodic with period 6, and since $100 \equiv 4 \bmod 6$, $200 \equiv 2 \bmod 6$, and $1998 \equiv 0 \bmod 6$, we have $1 = a_{100} = a_4 = -a_1$ and $0 = a_{200} = a_2$, and therefore $a_{1998} = a_6 = a_1 - a_2 = (-1) - 0 = -1$.

2. (a) How many subsets of a set with 8 elements have an odd number of elements?
(b) [MIT] How many 8 by 8 matrices are there in which each entry is 0 or 1 and each row and each column contains an odd number of 1's?

Solution: (a) Let N_0 and N_1 denote the number of subsets of an 8-element set with an even, respectively odd, number of elements. Clearly $N_0 + N_1 = 2^8$ and

$$N_0 = \sum_{\substack{k=0 \\ k \text{ even}}}^{8} \binom{8}{k}, \quad N_1 = \sum_{\substack{k=0 \\ k \text{ odd}}}^{8} \binom{8}{k}.$$

Moreover, by the binomial theorem,

$$N_0 - N_1 = \sum_{k=0}^{8} (-1)^k \binom{8}{k} = (1-1)^8 = 0,$$

so $N_0 = N_1$. Since $N_0 + N_1 = 2^8$, it follows that $N_0 = N_1 = 2^7 = 128$.
(b) The number of ways to fill a given row of 8 entries with 0's and 1's using an odd number of 1's is equal to the number of subsets of an 8-element set with an odd number of elements (as can be seen by labeling the entries with $1, 2, \ldots, 8$ and identifying a given row with the set of labels from $\{1, 2, \ldots, 8\}$ for which the corresponding entries are equal to 1. By part (a), this number is 2^7. The number of ways to fill the first 7 rows of the 8×8 matrix in this way is $(2^7)^7 = 2^{49}$. Each of these 2^{49} 7×8 matrices then has an odd number of 1's in each row, and there is exactly one way to add an 8th row to this matrix so that the number of 1's in each column is odd. Furthermore, if we choose the 8th row in this way,

then the total number of 1's in the matrix is even (since it is a sum of eight odd numbers), and since the number of 1's in the first 7 rows is odd (as a sum of 7 odd numbers), it follows that the number of 1's in the last row must be odd as well. Thus, all rows and all columns in this matrix have an odd number of 1's, and therefore the matrices with the requested properties are exactly the 2^{49} matrices constructed above.

3. [AIME 1988] For any positive integer k let $f_1(k)$ denote the sum of the squares of the digits of k (in decimal), and for $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1998}(11)$.

Solution: Iterating the map "sum of squares of the digits," we obtain the chain $11 \to 2 \to 4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 40 \to 16$, after which the sequence repeats itself with period 6. Hence $f_{1998}(11) = f_6(11) = 89$.

4. Let P(x) be a polynomial of degree n satisfying $P(k) = k$ for $k = 1, \ldots, n$ and $P(0) = 1$. Find $P(-1)$.

Solution: Let $Q(x) = P(x) - x$. Then $Q(x)$ is a polynomial of degree n whose roots are $k = 1, 2, \ldots, n$. Thus, $Q(x)$ is of the form $Q(x) = c\prod_{k=1}^{n}(x-k)$ for some constant c, and $P(x) = x + c\prod_{k=1}^{n}(x-k)$. Setting $x = 0$ gives $1 = P(0) = c(-1)^n n!$, so $c = (-1)^n/n!$. Hence

$$P(-1) = -1 + \frac{(-1)^n}{n!}\prod_{k=1}^{n}(-1-k) = -1 + (n+1) = n.$$

5. [MIT] Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers tending to 0 as $n \to \infty$, and let $b_n = a_n - 2a_{n+1} + a_{n+2}$. Assume that $b_n \geq 0$ for all n. Evaluate $\sum_{n=1}^{\infty} nb_n$.

Solution: Let $S_N = \sum_{n=1}^{N} nb_n$. For $N \geq 3$ we have

$$\begin{aligned}
S_N &= \sum_{n=1}^{N} nb_n = \sum_{n=1}^{N} na_n - 2\sum_{n=1}^{N} na_{n+1} + \sum_{n=1}^{N} na_{n+2} \\
&= \sum_{n=3}^{N} a_n(n - 2(n-1) + (n-2)) + a_1 + 2a_2 - 2a_2 - 2Na_{N+1} \\
&\qquad + (N-1)a_{N+1} + Na_{N+2} \\
&= a_1 - a_{N+1} - N(a_{N+1} - a_{N+2}).
\end{aligned}$$

By hypothesis, a_{N+1} tends to zero as $N \to \infty$, and we will show that

$$(*) \qquad \lim_{N\to\infty} N(a_{N+1} - a_{N+2}) = 0.$$

From $(*)$ it follows that $\sum_{n=1}^{\infty} nb_n = \lim_{N\to\infty} S_N = a_1$. To prove $(*)$, write $d_n = a_{n+1} - a_n$ and note that $b_n = d_n - d_{n+1}$. Since, by hypothesis, $b_n \geq 0$ for all n, the sequence $\{d_n\}_{n=1}^{\infty}$ is non-increasing. Hence $d_{N+1} \leq d_n$ for all $n \leq N+1$. Adding up these inequalities for the values $n = [N/2]+1, \ldots, N+1$, we obtain

$$(N/2)d_{N+1} \leq (N - [N/2])d_{N+1} \leq \sum_{n=[N/2]+1}^{N+1} d_n = (a_{[N/2]+1} - a_{N+2}).$$

Since $a_n \to 0$ as $n \to \infty$, the right-hand side here tends to zero as $N \to \infty$. Hence $(N/2)d_{N+1}$ tends to zero as $N \to \infty$, which is equivalent to $(*)$.
Remark: It is easy to find the answer a_1 by formally manipulating the series $\sum_{n=1}^{\infty} nb_n$, but a rigorous argument is much harder to come by. Infinite series cannot be split up and rearranged at will, so one needs to work with finite series like S_N and then take appropriate limits as was done above.