

UIUC Mock Putnam Exam 2/2000

October 16, 2000

Elementary Problems

- E1** Does there exist a power of 2 whose decimal representation ends in the digits 22? Explain!
- E2** Let x and y be real numbers greater than 1, let a denote the logarithm of x in base y and let b denote the logarithm of y in base x . Show that $a + b \geq 2$.
- E3** How many 6-digit integers are there whose digits are all distinct and occur in decreasing order (as in 965430)? (Hint: This problem has a simple elegant solution; don't try to solve it by brute force, by enumerating all cases.)
- E4** Prove that the product of any 100 consecutive positive integers is divisible by 100! .
- E5** Suppose that from every airport in Illinois a plane takes off and flies to the nearest neighboring airport. Assuming that all distances between airports are distinct, prove that there is no airport at which more than five planes land.

Advanced Problems

- A1** Show that there exist infinitely many powers of 7 whose decimal expansion ends in the digits 49.
- A2** Let x_0 and x_1 be two real numbers with $0 < x_1 \leq x_0 < 1$, and for $n \geq 2$ define x_n recursively by $x_n = x_{n-1}x_{n-2}$. Let $\phi = (1 + \sqrt{5})/2$. Show that the limit $\lim_{n \rightarrow \infty} x_{n+1}/x_n^\phi$ exists, and find its value.
- A3** Let f be a continuous, positive, decreasing function on $[0, 1]$. Show that

$$\int_0^1 f(x)(1 - 2x)dx \geq 0.$$

- A4** Given a nonnegative integer k , let S_k denote the sum of the infinite series $\sum_{n=1}^{\infty} n^k 2^{-n}$. Show that the numbers S_k are all integers.
- A5** A group of n people play a round-robin tournament (i.e., each player plays against every other player). Suppose that each game ends in a win or a loss (that is, draws are not allowed). Show that it is possible to label the n players P_1, P_2, \dots, P_n such that P_1 defeats P_2 , P_2 defeats P_3 , \dots , P_{n-1} defeats P_n .