

# UIUC Mock Putnam Exam 2/2000

October 16, 2000

## Elementary Problems

**E1** Does there exist a power of 2 whose decimal representation ends in the digits 22? Explain!

**Solution.** The answer is no, since a number ending with 22 must be of the form  $100k + 22 = 2(50k + 11)$  which is twice an odd number and therefore cannot be a power of 2 (other than  $2^1$  which does not end in digits 22).

**E2** Let  $x$  and  $y$  be real numbers greater than 1, let  $a$  denote the logarithm of  $x$  in base  $y$  and let  $b$  denote the logarithm of  $y$  in base  $x$ . Show that  $a + b \geq 2$ .

**Solution.** We have  $a = \log_y x = \log x / \log y$  and  $b = \log_x y = \log y / \log x = 1/a$ , so  $a + b = a + 1/a$ , and we have to show that the latter expression is at least 2. (Note that  $a$  is positive since  $x > 1$  and  $y > 1$ .) This can be done by calculus (differentiate the function  $f(x) = x + 1/x$  to find that  $f(x)$  is minimal when  $x = 1$ , and  $f(1) = 2$ ), or, more elegantly, by using the arithmetic-geometric mean inequality as follows:  $a + 1/a \geq 2\sqrt{a \cdot (1/a)} = 2$ .

**E3** How many 6-digit integers are there whose digits are all distinct and occur in decreasing order (as in 965430)? (Hint: This problem has a simple elegant solution; don't try to solve it by brute force, by enumerating all cases.)

**Solution.** The key is to note that there is a 1-1 correspondence between 6-digit sequences of distinct digits occurring in decreasing order, and 6-element subsets of the digit set  $\{0, 1, \dots, 9\}$ . Since there are  $\binom{10}{6}$  such subsets, the number of 6-digit integers with distinct, decreasing digits is  $\binom{10}{6}$ .

**E4** Prove that the product of any 100 consecutive positive integers is divisible by  $100!$ .

**Solution.** If  $P = n(n+1)\dots(n+99)$  is such a product, then  $P/100!$  is equal to the binomial coefficient  $\binom{n+99}{100}$ , which is an integer. Hence  $P$  is divisible by  $100!$ .

**E5** Suppose that from every airport in Illinois a plane takes off and flies to the nearest neighboring airport. Assuming that all distances between airports are distinct, prove that there is no airport at which more than five planes land.

**Solution.** Suppose, to get a contradiction, that there is an airport, say  $A$ , at which 6 (or more) planes land. Then for two of the six originating airports, say  $B$  and  $C$ , the angle formed by the routes from these cities to  $A$  is  $\leq 360/6 = 60$  degrees, i.e., in the triangle  $ABC$ , the angle at  $A$  is at most 60 degrees. But then a simple geometric argument shows that the sides  $BA$  and  $CA$  cannot both be smaller than  $BC$ . Hence,  $A$  cannot be the nearest neighbor to both  $B$  and  $C$ , which is a contradiction to our assumption.

## Advanced Problems

**A1** Show that there exist infinitely many powers of 7 whose decimal expansion ends in the digits 49.

**Solution.** We need to show that there are infinitely many positive integers  $n$  such that  $(1) 7^n \equiv 49 \pmod{100}$ . We will show that (1) holds whenever  $n$  is of the form  $n = 4m + 2$ , where  $m$  is a nonnegative integer. To see this, note first that  $7^4 = (49)^2 \equiv (-1)^2 = 1 \pmod{25}$  and  $7^4 \equiv (-1)^4 = 1 \pmod{4}$ , and so  $7^4 \equiv 1 \pmod{100}$ . Hence  $7^{4m+2} \equiv 1^m 7^2 = 49 \pmod{100}$ , as claimed.

**A2** Let  $x_0$  and  $x_1$  be two real numbers with  $0 < x_1 \leq x_0 < 1$ , and for  $n \geq 2$  define  $x_n$  recursively by  $x_n = x_{n-1}x_{n-2}$ . Let  $\phi = (1 + \sqrt{5})/2$ . Show that the limit  $\lim_{n \rightarrow \infty} x_{n+1}/x_n^\phi$  exists, and find its value.

**Solution.** Set  $q_n = x_n x_{n-1}^{-\phi}$ , so that we have to evaluate the limit  $\lim_{n \rightarrow \infty} q_n$ . A simple computation shows that  $\phi(1 - \phi) = -1$ . Using this relation and the given recurrence for  $x_n$ , we obtain  $q_n = x_{n-1}^{1-\phi} x_{n-2} = q_{n-1}^{1-\phi} x_{n-2}^{1+(1-\phi)\phi} = q_{n-1}^{1-\phi}$  for  $n \geq 2$ . Iterating this relation gives the explicit formula  $q_n = q_1^{(1-\phi)^{n-1}} = (x_1 x_0^{-\phi})^{(1-\phi)^{n-1}}$ . Since  $1 - \phi = (1 - \sqrt{5})/2$  is a number in the interval  $(-1, 0)$ , the exponent  $(1 - \phi)^{n-1}$  tends to 0 as  $n \rightarrow \infty$ , and so  $q_n$  tends to 1 as  $n \rightarrow \infty$ . Hence the limit in question exists and equals 1.

**Remark:** Setting  $y_n = \ln x_n$  and taking logarithms transforms the given recurrence into  $y_n = y_{n-1} + y_{n-2}$ , which is the recurrence for the Fibonacci numbers. The general solution to that recurrence is of the form  $y_n = c_1 \phi^n + c_2 (1 - \phi)^n$ , with constants  $c_1$  and  $c_2$ . From this formula, it is easy to see that  $\lim_{n \rightarrow \infty} (y_{n+1} - \phi y_n) = 0$ , which is equivalent to the asserted relation for  $x_n$ .

**A3** Let  $f$  be a continuous, positive, decreasing function on  $[0, 1]$ . Show that

$$\int_0^1 f(x)(1 - 2x) dx \geq 0.$$

**Solution.** Splitting the range of integration into the two parts  $0 \leq x \leq 1/2$  and  $1/2 \leq x \leq 1$  and making the change of variables  $y = 1 - x$  in the integral over the latter range, the given integral can be written as

$$\int_0^{1/2} f(x)(1 - 2x) dx + \int_0^{1/2} f(1 - y)(2y - 1) dy = \int_0^{1/2} (f(x) - f(1 - x))(1 - 2x) dx.$$

Since  $f$  is decreasing, we have  $f(x) - f(1 - x) \geq 0$  for  $0 \leq x \leq 1/2$ . Hence the integrand in the last integral is nonnegative in the range of integration, and the integral is therefore nonnegative as well.

**Remark:** The assumption that  $f(x)$  is positive was not really needed here; the result remains true for any decreasing continuous function  $f$ .

**A4** Given a nonnegative integer  $k$ , let  $S_k$  denote the sum of the infinite series  $\sum_{n=1}^{\infty} n^k 2^{-n}$ . Show that the numbers  $S_k$  are all integers.

**Solution.** For  $k = 0$  the given sum is a geometric series summing to 1, so it remains to consider the case when  $k \geq 1$ . To this end, we will relate the numbers  $S_k$  to the derivatives  $T_k = f^{(k)}(1/2)$  of the function  $f(x) = \sum_{n=0}^{\infty} x^n = (1 - x)^{-1}$ . We have  $f^{(k)}(x) = k!(1 - x)^{-k-1}$ , and so the

numbers  $T_k = f^{(k)}(1/2) = k!2^{k+1}$  are all integers. On the other hand, differentiating the series  $f(x) = \sum_{n=0}^{\infty} x^n$  termwise, we see that

$$T_k = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)(1/2)^{n-k} = \sum_{n=0}^{\infty} P_k(n)(1/2)^n$$

where

$$P_k(n) = n(n+1)\cdots(n+k-1) = n^k + \sum_{i=0}^{k-1} a_{i,k}n^i$$

is a polynomial in  $n$  of degree  $k$  with integer coefficients  $a_{i,k}$ . Hence, for each  $k$ , we have

$$T_k = S_k + \sum_{i=0}^{k-1} a_{i,k}S_i$$

(where, for convenience, we have extended the summation in the definition of  $S_0$  to include the  $n = 0$  term, so that  $S_0 = \sum_{n=0}^{\infty} n^k 2^{-n} = 2$ ). Since the numbers  $T_k$  and the coefficients  $a_{i,k}$  are integers, it follows from this relation that, if the numbers  $S_i$ ,  $0 \leq i \leq k-1$ , are all integers, then so is  $S_k$ . Since  $S_0 = 2$  is an integer, it follows by strong induction that all numbers  $S_k$  are integers, as claimed.

**A5** A group of  $n$  people play a round-robin tournament (i.e., each player plays against every other player). Suppose that each game ends in a win or a loss (that is, draws are not allowed). Show that it is possible to label the  $n$  players  $P_1, P_2, \dots, P_n$  such that  $P_1$  defeats  $P_2$ ,  $P_2$  defeats  $P_3$ ,  $\dots$ ,  $P_{n-1}$  defeats  $P_n$ .

**Solution.** We use induction on  $n$ . For  $n = 2$ , the result holds trivially, so suppose  $n \geq 3$  and that the claim holds for  $n-1$  players. Pick one of the  $n$  players, say  $P$ , and use the induction hypothesis to label the remaining  $n-1$  players  $P_1, P_2, \dots, P_{n-1}$ , so that  $P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_{n-1}$  where  $A \rightarrow B$  means that  $A$  defeats  $B$ . If  $P$  defeats  $P_1$ , we can tack on  $P$  at the beginning of the sequence  $P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_{n-1}$  to get a sequence of the required form involving all  $n$  players. Similarly, if  $P$  loses to  $P_{n-1}$ , tacking on  $P$  at the end of this sequence yields the result. Assume therefore that  $P$  loses to  $P_1$ , but defeats  $P_{n-1}$ , and let  $k$  be the minimal index such that  $P$  loses to  $P_1, \dots, P_{k-1}$ , but defeats  $P_k$ . Then the sequence  $P_1 \rightarrow \cdots \rightarrow P_{k-1} \rightarrow P \rightarrow P_k \rightarrow P_{k+1} \rightarrow \cdots \rightarrow P_{n-1}$  is a sequence of the desired form involving all  $n$  players. This completes the induction.