UIUC Mock Putnam Exam 2/2001

October 23, 2001

Solutions

1 You are standing at the ocean's edge awaiting the arrival of a sailing ship having a 60 foot high mast. When you first see the tip of the mast on the horizon, about how far away is the ship? You may assume the earth's radius to be 4000 miles and that your eyes are at height 7 feet above the water.

Solution. We estimate the sea distance by the straight line distance d between the eye and the tip of the mast. Consider three concentric circles in the plane, $\mathcal{C}_1$ of radius 4000 (mi.), $\mathcal{C}_2$ of radius $4000+\delta$, $\delta = 7/5280$, and $\mathcal{C}_3$ of radius $4000+\Delta$, $\Delta = 60/5280$. Let T be the tangent line to $\mathcal{C}_1$ (the line of sight from the eye, tangent to the earth, to the top of the mast) and t be the point of tangency. Let e be a point of intersection of T with $\mathcal{C}_2$ (the location of the eye), and l the distance from the t to e. Let m be the point of intersection of T with $\mathcal{C}_3$ farthest from e, (the location of the top of the mast), and let L be the distance from the t to m. We require $L + l$.
By Pythagoras' theorem, neglecting δ^2 and Δ^2, we have

$$l^2 = (4000+\delta)^2 - 4000^2 \approx 8000\delta, \qquad L^2 = (4000+\Delta)^2 - 4000^2 \approx 8000\Delta.$$

Thus

$$l \approx \sqrt{\frac{8000 \cdot 7}{5280}} = 10\sqrt{\frac{7}{66}}, \qquad L \approx \sqrt{\frac{8000 \cdot 60}{5280}} = 10\sqrt{10/11}.$$

To get real numbers, we can approximate L and l by the iterative square root method. Taking $l_0/10 = 1/3$ and $L_0/10 = 1$ we get

$$l_1/10 = \frac{1}{2}\Big(\frac{1}{3} + \frac{7/66}{1/3}\Big) = \frac{43}{132} \approx .3257, \qquad L_1/10 = \frac{1}{2}\Big(1 + \frac{10}{11}\Big) = \frac{21}{22} \approx .9545.$$

Thus the total distance is about $3.26 + 9.55 \approx 12.8$ miles.

2 Show that $7^{1/3} + 9^{1/3} < 4$.

Solution. $f(x) := x^{1/3}$ is a concave (=convex down) function for $x > 0$, so $f(8) > (f(7) + f(9))/2$, which is equivalent to the assertion.

3 Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2n^2 - n}.$$

Solution. Let S_N denote the Nth partial sum of the given series. Then

$$S_N = \sum_{n=1}^{N} \frac{2}{n(2n-1)} = 2\sum_{n=1}^{N}\left(\frac{1}{2n-1} - \frac{1}{2n}\right) = 2\sum_{n=1}^{2N} \frac{(-1)^{n+1}}{n}.$$

The last sum is an alternating series with decreasing terms, so it converges, as $N \to \infty$, to the infinite series $(*)$ $\sum_{n=1}^{\infty}(-1)^{n+1}/n = 1-(1/2)+(1/3)-(1/4)+\cdots$. We will show that the series $(*)$ is equal to $\ln 2$, so that the given series has sum $2\ln 2$. To evaluate $(*)$, note that the Nth partial sum of $(*)$ is

$$\begin{aligned}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n} &= \sum_{n=0}^{N-1}\int_0^1(-x)^n dx = \int_0^1\sum_{n=0}^{N-1}(-x)^n dx\\ &= \int_0^1\frac{1-(-x)^N}{1+x}dx = \ln 2 + (-1)^{N+1}\int_0^1\frac{x^N}{1+x}dx.\end{aligned}$$

The latter integral is bounded by $\int_0^1 x^N dx = 1/(N+1)$ and hence tends to 0 as $N \to \infty$. Thus, the partial sums above converge to $\ln 2$. (Formally, one could evaluate this series using the Taylor expansion of $\ln(1+x)$ at $x = 1$; however, the fact that the expansion is valid at $x = 1$ is not obvious and requires some justification.)

4 Suppose every point of the plane has been colored either red, or blue. Prove that, for one of these colors, there exist pairs of points at *every* mutual distance.

Solution. If the conclusion does not hold, then there exist positive real numbers a and b such that no two red points have distance a and no two blue points have distance b. Without loss of generality we may assume $a \geq b$. Pick a red point (we may assume there exist red points since otherwise the blue points have the desired property), and draw a circle of radius a, centered at this point. By our assumption, none of the points on the circle can be red, so all points on the circle must be blue. But then, since $b < a$, we can find two (blue) points on the circle that have distance b, contradicting our assumption.

5 Let $f(n) = [n + \sqrt{n} + 1/2]$, where $[x]$ denotes the greatest integer $\leq x$. Determine, with proof, the set of positive integers m that can be expressed in the form $m = f(n)$ for some integer n.

Solution. We show that the set in question consists of all positive integers except the squares. To see this, let $f_0(x) = x+\sqrt{x}+1/2$, so that $f(n) = [f_0(n)]$. Hence, $m = f(n)$ if and only if $m \leq f_0(n) < m+1$. Since $f_0(n)$ is an increasing function of n, it follows that an integer m is **not** of the form $m = f(n)$ if and only if there exists an integer n such that (1) $f_0(n) < m$ and (2) $f_0(n+1) \geq m+1$. Since $f_0(n+1) = n+1+\sqrt{n+1}+1/2$, (2) is equivalent to $m-n \leq \sqrt{n+1}+1/2$, whereas (1) holds if and only if $\sqrt{n}+1/2 < m-n$. Now (1) and (2) imply $\sqrt{n} < (m-n)-1/2 \leq \sqrt{n+1}$, and upon squaring we get $n < (m-n)^2-(m-n)+1/4 \leq n+1$. Since $(m-n)^2-(m-n)$ is an integer, this forces $(m-n)^2-(m-n)$ to be equal to n, i.e., $(m-n)^2 = (m-n)+n = m$. Thus, (1) and (2) imply that m is a square. Conversely, if m is a (positive) square, say $m = h^2$, then setting $n = m-h$, we have $f_0(n) = (m-h)+\sqrt{m-h}+1/2 = m-h+\sqrt{h^2-h}+1/2$, and since $\sqrt{h^2-h} < \sqrt{h^2-h+1/4} = h-1/2$, it follows that $f_0(n) < m$, i.e., (1) holds. Similarly, $f_0(n+1) = (m-h+1)+\sqrt{h^2-h+1}+1/2 > (m-h+1)+\sqrt{h^2-h+1/4}+1/2 = m+1$, so (2) holds as well.

6 Find the precise set of numbers x for which the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+\sqrt{n})^x}$$

converges.

Solution. We show that the series converges for all $x > 3$ and diverges for all $x \le 3$. To prove the latter part, it clearly suffices to show that the series diverges at $x = 3$. Now, for $m \le \sqrt{n}$, we have $m + \sqrt{n} \le 2\sqrt{n}$, and the number of such m is $[\sqrt{n}] \ge \sqrt{n} - 1$ so the given double series is bounded from below by

$$\sum_{n=1}^{\infty}\sum_{m\le\sqrt{n}}\frac{1}{(2\sqrt{n})^3)} \ge \frac{1}{8}\sum_{n=1}^{\infty}\frac{\sqrt{n}-1}{n^{3/2}},$$

which diverges by comparison with the harmonic series.

Now suppose that $x > 3$. We split the given series into two parts, say S_1 and S_2, where S_1 contains those terms in which $m \le \sqrt{n}$ and S_2 those in which $m > \sqrt{n}$, i.e., $n < m^2$. To estimate S_1, we use the inequality $(m + \sqrt{n})^{-1} \le 1/\sqrt{n}$ to get

$$S_1 \le \sum_{n=1}^{\infty}\sum_{m=1}^{[\sqrt{n}]}\frac{1}{(\sqrt{n})^x} \le \sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{n}^x} = \sum_{n=1}^{\infty}\frac{1}{n^{(x-1)/2}},$$

which converges since $(x-1)/2 > 1$. Similarly, using the bound $(m + \sqrt{n}))^{-1} \le 1/m$ we get

$$S_2 \le \sum_{m=1}^{\infty}\sum_{n=1}^{m^2-1}\frac{1}{m^x} \le \sum_{m=1}^{\infty}\frac{m^2}{m^x} = \sum_{m=1}^{\infty}\frac{1}{m^{x-2}},$$

which also converges. Hence the entire series converges when $x > 3$.