Hints and Answers to Mock Putnam Exam 2

November 11, 1996

1. The answer is $S = (-1)^r\binom{n-1}{r}$. This can be proved by induction on r. Alternatively, applying the recursion formula for binomial coefficients $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$ ($i = 1, \ldots, n-1$) produces a telescoping sum that simplifies to the above expression for S.

2. This was a problem in the first Putnam Competition. The answer is $3a\sqrt{3}$. The proof is a tedious and rather uninspiring calculation.

3. This was a 1971 Putnam problem. The only polynomial with the given property is the polynomial $P(x) = x$. To prove this, show by induction that $P(k^2+1) = k^2+1$ for $k = 0, 1, 2, \ldots$ Thus, $P(x)$ agrees with the polynomial x on infinitely many values, and the two polynomials must therefore be equal.

4. This problem was incorrectly stated. The correct condition on the a_i should have been that if any term a_i is removed the remaining $2n$ terms can be divided into two sets *of equal cardinality* and having equal sums. To show that this is possible only if all a_i's are equal, one first observes that the a_i's must have the same parity, i.e., they must either be all even or all odd. Next, if a set $\{a_i : i = 1, \ldots, 2n+1\}$ has the stated property, then so does any translate of this set, i.e., the sets $\{a_i + k : i = 1, \ldots, 2n+1\}$, where k is a fixed integer. Moreover, if all a_i's are even, the set $\{a_i/2 : i = 1, \ldots, 2n+1\}$ has the same property. Starting with a given set $\{a_i\}$ and using these two operations (translation by 1 and division by 2), one eventually ends up with a set consisting of all 0's. This is only possible if the original a_i's had all been equal.

5. This was a Putnam problem in 1980. If $P(t)$ is a non-zero constant polynomial, then a direct integration gives the result. We may therefore assume that $P(t)$ has degree at least 1.
   Let $f(x) = \int_0^x P(t)e^{it}dt$. Then $f(x) = 0$ if and only if the two integrals $\int_0^x P(t)\sin t dt$ and $\int_0^x P(t)\cos t dt$ are simultaneously zero. By induction, one can show that for each nonnegative integer k, $\int_0^x t^k e^{it}dt = Q_k(x)e^{ix} + c_k$, where $Q_k(x)$ is a polynomial of degree k with real coefficients and c_k a constant. It follows that, if $P(t) = \sum_{k=0}^n a_k t^k$ is a polynomial of degree n with real coefficients, then $f(x) = Q(x)e^{ix} + c$ where $Q(x) = \sum_{k=0}^n a_k Q_k(x)$ is a polynomial of the same degree as $P(t)$ with real coefficients and $c = \sum_{k=0}^n a_k c_k$ a constant. Setting $x = 0$, we see that c must be equal to $Q(0)$. Thus, $f(x) = 0$ is equivalent to $(*)$ $Q(x) = -Q(0)e^{-ix}$, and we have to show that $(*)$ has only finitely many solutions.
   By our assumption, the polynomials $P(x)$ and $Q(x)$ are non-constant. Hence $|Q(x)| \to \infty$ if $|x| \to \infty$. Therefore, all solutions to $(*)$ must fall into an interval of the form $|x| \le C$, where C is a constant.
   If $Q(0) = 0$, then $(*)$ reduces to the equation $Q(x) = 0$ which has at most n solutions by the fundamental theorem of algebra, and we are done. Suppose therefore that $Q(0) \neq 0$. Taking real parts, we obtain from $(*)$ $Q(x) = -Q(0)\cos x$. If this equation had infinitely many solutions in the interval $[-C, C]$, then by applying Rolle's theorem to the function $Q(x)+Q(0)\cos x$ and its successive derivatives, we obtain that, for each positive integer k, the equation $(**)$ $Q^{(4k)}(x) = -Q(0)\cos x$, also had infinitely many solutions x in the same interval. (Note that Rolle's theorem does not hold for complex-valued functions; thus it was necessary to take real parts in $(*)$ before applying the theorem.) However, since $Q(x)$ is a polynomial of degree n, the left side of $(**)$ is zero if $k > n/4$, whereas (since $Q(0) \neq 0$) the right side can only be zero at the zeroes of $\cos x$, of which there are only finitely many in the interval $[-C, C]$. Hence $(*)$ can have only finitely many solutions.

6. Using the identity $2\sin A\sin B = \cos(A-B) - \cos(A+B)$ one obtains a telescoping series with sum $(1-\cos x)/2$.