

Mock Putnam Exam 3

November 16, 1996

Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different. (J. W. von Goethe)

1. In how many ways can 10,000 be expressed as a product of three positive integers, if factorizations which differ only in the order of factors are not considered different?
2. Into how many regions do n lines divide the plane, assuming no two lines are parallel and no three lines intersect in the same point?
3. Among all powers of 2, what percentage begin with the digit 1 in their decimal representation? More precisely, if $f(n)$ denotes the number of integers among the first n powers of 2 (i.e., $2^1, 2^2, \dots, 2^n$) whose decimal representation begins with the digit 1, show that the limit $\lim_{n \rightarrow \infty} f(n)/n$ exists and compute its value.
4. Show that for any positive integer n not equal to 2, 3, or 5, a square can be decomposed into n non-intersecting squares. An example of such a decomposition when $n = 6$ is sketched below.
5. Every point in the plane is colored either red, yellow, or blue. Prove that there are two points of the same color having mutual distance 1.
6. A certain country has finitely many cities. Any pair of these cities is connected by a road. However, all roads in this country are one-way roads, and it is therefore not always possible to travel from one city to another city. Show that the country has a city ("capital") that can be reached from every other city either directly or via exactly one intermediate city. (Hint: Try induction on the number of cities.)