# UIUC Department of Mathematics

# Solutions to Mock Putnam Exam 3

## October 27, 1997

1.. Let $P(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree $n \geq 1$ with real coefficients. Show that if $\sum_{i=0}^{n} a_i/(i+1) = 0$ then $P(x)$ has at least one real root.

**Solution.** Since $\sum_{i=0}^{n} a_i/(i+1) = \int_0^1 P(x)dx$ and $P(x)$, as a polynomial, is a continuous function, the mean value theorem for integrals implies that $P(x) = 0$ for some $x \in (0,1)$.

2.. Nine mathematicians meet at an international conference and discover that, among any three of them, at least two speak a common language, and that none of them speaks more than three languages. Prove that there are at least three mathematicians who can speak the same language.

**Solution.** Label the mathematicians by $1, 2, \ldots, 9$, and let $S_i$ denote the set of mathematicians (including $i$) that speak a common language with mathematician $i$. For each $j \in S_i$ there thus exists a language $L_{ij}$ that both $i$ and $j$ speak. If, for some $i$, $L_{ij_1} = L_{ij_2}$ for two distinct elements $j_1, j_2 \in S_i$, both different from $i$, then the three mathematicians $i$, $j_1$, and $j_2$, all speak the language $L_{ij_1}$, and we are done. Otherwise we have that for each $i$, the languages $L_{ij}$, for $j \in S_i$, $j \neq i$, are distinct. We will show by contradiction that this case cannot occur. Since no mathematician speaks more than three languages, it follows in this case that the sets $S_i$ can have at most four elements. Pick $i \notin S_1$ (there are at least 5 such $i$ since $|S_1| \leq 4$), and then pick $j \notin S_1 \cup S_i$ (there is at least one such $j$ since $|S_1 \cup S_i| \leq 8$). Then 1, $i$, $j$ are all distinct, and since $i, j \notin S_1$ and $j \notin S_i$, no two of these three mathematicians speak the same language. This contradicts the assumption that among any three mathematicians at least two speak the same language. The assertion is therefore proved.

3.. Suppose each point in the plane is colored either orange or blue. Prove that there exists a color (either orange or blue) such that, for every positive distance $d$, there exists a pair of points of that color having distance $d$.

**Solution.** Suppose, to the contrary, that there exist positive distances $d_o$ and $d_b$ such that no two orange points have distance $d_o$ and no two blue points have distance $d_b$. Without loss of generality, we may assume $d_o \leq d_b$. Our assumption implies, in particular, that there exists at least one blue point, say $B$. Now draw a circle of radius $d_b$ around $B$. Since no two blue points have distance $d_b$, every point on the circle must be colored orange. Since $d_o \leq d_b$, there exist two points on the circle that are distance $d_o$ apart. This contradicts the assumption that no two orange points have distance $d_o$. Hence, at least one of the two colors contains pairs of points at every distance.

4.. Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let $b_1, b_2, \ldots, b_n$ be a permutation of the $a_i$'s . Show that $\sum_{i=1}^n a_i/b_i \geq n$.

**Solution.** By the arithmetic-geometric mean inequality (a.m. $\geq$ g.m.) we have

$$\frac{1}{n}\sum_{i=1}^{n}\frac{a_i}{b_i} \geq \left(\prod_{i=1}^{n}\frac{a_i}{b_i}\right)^{1/n} = 1,$$

from which the desired inequality follows.

5.. Show that $2^{-x} + 2^{-1/x} \leq 1$ for all positive real values of $x$.

**Solution.** Let $f(x) = 2^{-x} + 2^{-1/x}$. Since $f(x) = f(1/x)$ and $f(1) = 1$, it suffices to show that, for $0 < x < 1$, $f(x) \leq 1$. Since $f(x)$ has a continuous derivative on $(0, \infty)$ and tends to 1 if $x \to 1-$ or $x \to 0+$, the maximum value of $f(x)$ on the interval $(0, 1]$ is either is to 1 or equal to the value of $f(x)$ at an interior point $x \in (0, 1)$ at which $f'(x) = 0$. Thus, to show that $f(x) \leq x$ for all points $x \in (0, 1]$ it suffices to show that $f(x) \leq x$ *for all such points that also satisfy* $f'(x) = 0$.

A simple calculation shows that $f'(x) = 0$ if and only if $2^{-1/x} = x^2 2^{-x}$. Replacing $2^{-1/x}$ in the definition of $f(x)$ by $x^2 2^{-x}$, we see that the value of $f(x)$ at any point with $f'(x) = 0$ is equal to $2^{-x}(1 + x^2)$. Hence, to prove that this value is $\leq 1$, it suffices to show that the function $g(x) = 2^{-x}(1 + x^2)$ satisfies $g(x) \leq 1$ for $x \in (0, 1)$. Now, $g'(x) = 2^{-x} h(x)$, with $h(x) = 2x - (\ln 2)(1 + x^2)$. Since $h'(x) = 2 - 2(\ln 2)x > 0$ for $x \in (0, 1)$, and $h(0) = -\ln 2 < 0$, $h(1) = 2 - 2\ln 2 > 0$, the function $h$ is strictly increasing and has a unique zero $x_1 \in (0, 1)$. Hence $g(x)$ is decreasing for $0 \leq x \leq x_1$ and increasing for $x_1 < x \leq 1$. This implies that for all $x \in [0, 1]$, $g(x) \leq \max(g(0), g(1)) = 1$, which proves our claim.