UIUC Department of Mathematics

Mock Putnam Exam 3

November 9, 1998

Solutions

1. [Barbeau, #230] Evaluate $f(n) = 1^2 - 2^2 + 3^2 - \cdots + (2n-1)^2 - (2n)^2$.

Solution: Since $(2k-1)^2 - (2k)^2 = 1 - 4k$, the given sum is

$$f(n) = \sum_{k=1}^{n}(1-4k) = n - 4\frac{n(n+1)}{2} = -2n^2 - n.$$

2. Let F_n be the nth Fibonacci number, i.e., $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Show that $F_n^2 + F_{n+1}^2 = F_{2n+1}$ for all $n \geq 1$.

Solution: From the recurrence relation for F_n we see that $F_{n+2} = F_{n+1} + F_n$, $F_{n+3} = F_{n+2} + F_{n+1} = 2F_{n+1} + F_n$, $F_{n+4} = F_{n+3} + F_{n+2} = 3F_{n+1} + 2F_n$, and, by induction, $F_{n+m} = F_m F_{n+1} + F_{m-1}F_n$ for all positive integers m and n. Setting $m = n+1$ gives the desired relation.

3. [Putnam 1986] Determine the rightmost digit (in decimal) of $\left[\frac{10^{20000}}{10^{100}+3}\right]$.

Solution: Let x denote the number in brackets. Expanding $(1 + 3 \cdot 10^{-100})^{-1}$ into a geometric series, we obtain

$$x = \sum_{n=0}^{\infty}(-1)^n 3^n 10^{19,900-100n}.$$

In the last sum, all terms with $n < 199$ are all divisible by 10 and the term $n = 199$ equals $(-3)^{199}$. Also, since the series is alternating with decreasing terms, the sum of the terms with $n \geq 200$ is positive and bounded from above by the first of these terms, i.e., $3^{200}10^{-100}$, which is less than 1. Thus, the last digit of $[x]$ is equal to the last digit of $N = (-3)^{199}$ or, equivalently, the residue of N modulo 10. Since $(-3)^4 \equiv 1$ modulo 10, we have $(-3)^{199} \equiv (-3)^3 \equiv 3$ modulo 10, so the rightmost digit of $[x]$ is 3.

4. [Putnam 1984] Express

$$\sum_{k=1}^{\infty}\frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)}$$

as a rational number.

Solution: Let S denote the given infinite series, and let a_k denote the kth term in this series. Then

$$a_k = \frac{(2/3)^k}{3(1-(2/3)^{k+1})(1-(2/3)^k)} = \frac{1}{1-(2/3)^k} - \frac{1}{1-(2/3)^{k+1}}.$$

Thus, the given series "telescopes," and its partial sums are given by

$$\sum_{k=1}^{n} a_k = \frac{1}{1-(2/3)} - \frac{1}{1-(2/3)^{n+1}}.$$

Letting $n \to \infty$, the last expression tends to 2. Thus, $S = 2$.

5. [Newman, #87] Let $\{a_n\}$ be a sequence of positive real numbers satisfying $a_n < a_{n+1} + a_{n^2}$ for all n. Show that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Solution: Define sets of indices I_k recursively by $I_0 = \{2\}$, and $I_k = (I_{k-1}+1) \cup I_{k-1}^2$ for $k \geq 1$ (where $A^2 = \{a^2 : a \in A\}$ and $A + 1 = \{a+1 : a \in A\}$ for any set of integers A). By induction we see that the smallest element in I_k is $k+2$. Assume for the moment that *for every k, the sets $I_k + 1$ and I_k^2 are disjoint.* The given inequality $a_n < a_{n+1} + a_{n^2}$ then implies

$$\sum_{i \in I_{k=1}} a_i < \sum_{i \in I_{k-1}+1} a_i + \sum_{i \in I_{k-1}^2} a_i = \sum_{i \in I_k} a_i$$

for every $k \geq 1$, and by induction it follows that

$$0 < a_2 = \sum_{i \in I_0} a_i < \sum_{i \in I_k} a_i \leq \sum_{i=k+2}^{\infty} a_i$$

for all $k \geq 1$. Thus, the tails of the series $\sum_{i=1}^{\infty} a_i$ do not converge to zero, and the series therefore diverges.

It remains to prove the above claim that the sets $I_k + 1$ and I_k^2 are disjoint for every k. Suppose this is not the case, and let k denote the smallest index such that $I_k + 1$ and I_k^2 have a common element, say m^2. Then $m \geq k+2$ since $m \in I_k$ and the smallest element of I_i is $k+2$. Also, since $m^2 - 1 \in I_k = (I_{k-1}+1) \cup I_{k-1}^2$, and $m^2 - 1$ is not a square, it follows that $m^2 - 1 \in I_{k-1} + 1$, i.e., $m^2 - 2 \in I_{k-1}$. Continuing in this way, we see that $m^2 - 3 \in I_{k-2}, \ldots, m^2 - 2(m-1) \in I_{k-2m+3}$, since none of the numbers $m^2-2, \ldots, m^2-2(m-1)$ are squares. Thus, $k-2m+3 \geq 0$, or $m \leq (k+3)/2$, which contradicts the inequality $m \geq k+2$. Hence the claim is proved.