UIUC Department of Mathematics

Mock Putnam Exam 3

October 27, 1999

This exam is intended as a practice test for the real Putnam Exam and will be graded in the same way. To receive credit, you need to explain yourself clearly and succinctly; an answer alone won't do.

Graded exams will be returned at next Monday's Putnam Training Session.

Solutions will be posted by the end of this week at
http://www.math.uiuc.edu/~hildebr/putnam/mockputnam.html.

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Problem 1. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be $2n$ positive real numbers. Show that at least one of the inequalities

$$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \geq n \tag{1}$$

or

$$\frac{b_1}{a_1} + \cdots + \frac{b_n}{a_n} \geq n \tag{2}$$

holds.

Problem 2. Let $a_1, a_2, \ldots, a_{10}$ and $b_1, b_2, \ldots, b_{10}$ be two permutations of the numbers $1, 2, \ldots, 10$. Show that the products $a_1b_1, a_2b_2, \ldots, a_{10}b_{10}$ cannot be all distinct modulo 11.

Problem 3. (Putnam 1982) Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx.$$

Problem 4. Show that there exists an infinite set of integers of the form $2^n - 3$ with the property that no two elements in this set have a common prime factor.

Problem 5. Let $\pi(n)$ denote the number of primes less than or equal to n. Show that there are infinitely many positive integers n that are divisible by $\pi(n)$.