UIUC Department of Mathematics

Mock Putnam Exam 3 Solutions

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Problem 1. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be $2n$ positive real numbers. Show that at least one of the inequalities

$$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \geq n \tag{1}$$

or

$$\frac{b_1}{a_1} + \cdots + \frac{b_n}{a_n} \geq n \tag{2}$$

holds.

Solution.
Method I: It is easy to check that the function $f(x) = x + 1/x$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, so that $f(x) \geq f(1) = 2$ for all $x > 0$. Setting $x = a_i/b_i$ and adding the resulting inequalities for $i = 1, 2, \ldots, n$, shows that the sum of the two expressions on the left of (1) and (2) is at least $2n$. Hence one of these expressions must be at least n.
Method II: Suppose without loss of generality that $a_1 a_2 \ldots a_n \geq b_1 b_2 \ldots b_n$. Then the geometric mean of the numbers $a_1/b_1, \ldots, a_n/b_n$ is at least 1. Since the arithmetic mean of these numbers is $1/n$ times the left-hand side of (1), and the geometric mean is always less than or equal to the arithmetic mean (by the AGM inequality), (1) holds.
Method III: By the Cauchy-Schwarz inequality,

$$n^2 = \left(\sum_{i=1}^{n} \sqrt{a_i/b_i} \cdot \sqrt{b_i/a_i} \right)^2 \leq \left(\sum_{i=1}^{n} \frac{a_i}{b_i} \right) \left(\sum_{i=1}^{n} \frac{b_i}{a_i} \right)$$

Hence one of the sums on the left of (1) and (2) must be $\geq n$.

Problem 2. Let $a_1, a_2, \ldots, a_{10}$ and $b_1, b_2, \ldots, b_{10}$ be two permutations of the numbers $1, 2, \ldots, 10$. Show that the products $a_1 b_1, a_2 b_2, \ldots, a_{10} b_{10}$ cannot be all distinct modulo 11.

Solution.
We have $P = \prod_{i=1}^{10} a_i = \prod_{i=1}^{10} b_i = 10!$, so $\prod_{i=1}^{10} (a_i b_i) = P^2 = 10!^2$. If the numbers $a_i b_i$ $(i = 1, 2, \ldots, 10)$ were all distinct modulo 11, then $P^2 \equiv P$ or $10!(10! - 1) \equiv 0$ modulo 11. Since 11 has no common factors with $10!$, the latter relation implies $10! \equiv 1$ modulo 11. However, a simple calculation (or Wilson's theorem) shows that $10! \equiv -1$ modulo 11.

Problem 3. (Putnam 1982) Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx.$$

Solution.
The given integral is the limit, as $T \to \infty$ of the integral

$$I(T) = \int_0^T \frac{\arctan(\pi x) - \arctan(x)}{x} dx = \int_0^{\pi T} \frac{\arctan y}{y} dy - \int_0^T \frac{\arctan x}{x} dx$$
$$= \int_T^{\pi T} \frac{\arctan x}{x} dx.$$

Since $\arctan x \le \pi/2$ for positive x and $\arctan x \to \pi/2$ as $x \to \infty$, for any given $\epsilon > 0$, the latter integral falls between $(\pi/2 - \epsilon)$ and $\pi/2$ times the integral $\int_T^{\pi T}(1/x)dx = \log \pi$, provided T is sufficiently large in terms of ϵ. Hence, $I(T)$ converges to $(\pi/2)\log \pi$ as $T \to \infty$, and the given integral is equal to $(\pi/2)\log \pi$.

Problem 4. Show that there exists an infinite set of integers of the form $2^n - 3$ with the property that no two elements in this set have a common prime factor.

Solution.
It is enough to show that given any finite set S of such integers, there exists an n such that $2^n - 3$ has no common prime factors with any of the integers in S. To see this, let S be such a set, and let $p_1, p_2, \dots, p_k$ be the distinct prime factors of the elements of S. Let $n = 1 + (p_1 - 1)\dots(p_k - 1)$. Since, by Fermat's theorem, $2^{p_i - 1} \equiv 1$ modulo p_i, we have $2^n \equiv 2 \cdot 2^{(p_1-1)\dots(p_k-1)} \equiv 2 \cdot 1$ modulo p_i and so $2^n - 3 \equiv -1$ modulo p_i for each of the primes p_i. Hence $2^n - 3$ is not divisible by any of the primes p_i, as claimed.

Problem 5. Let $\pi(n)$ denote the number of primes less than or equal to n. Show that there are infinitely many positive integers n that are divisible by $\pi(n)$.

Solution.
We will show more precisely that, for every integer $m \ge 2$, there exists n such that $n = m\pi(n)$. Let $m \ge 2$ be given. Since $\pi(2)/2 = 1$ and $\pi(n)/n \to 0$ as $n \to \infty$, there exists a maximal integer k such that $\pi(mk)/(mk) \ge 1/m$, or equivalently $\pi(mk) \ge k$. If $\pi(mk) = k$, we are done. However, if $\pi(mk) > k$, then $\pi(m(k+1)) \ge \pi(mk) \ge k + 1$, contradicting the maximality of k.