UIUC Mock Putnam Exam 3/2001

November 13, 2001

Name:

Rules: No calculators, books, notes, etc. Show all work, but do not hand in scratch work.

Grading: Problems will be graded in much the same way as the Putnam exam problems. Each problem is worth 10 points. Partial credit may be given, but only if significant progress towards a solutions is being made. As in the Putnam exam, a clear, logical write-up of the solution is essential. Exam solutions will be posted on the Math Contest Web Page (www.math.uiuc.edu/contests.html) within a few days.

Time: Two hours. If you do four problems, this works out to 30 minutes per problem, the same amount of time per problem you get in the Putnam exam (which has 12 problems, to be solved in 6 hours). The problems have not been grouped into elementary and advanced problems, but they are roughly arranged in increasing order of difficulty. **Circle those problems on the coversheet (this sheet) that you want to have graded.**

Scratch work: Do all scratch work on separate sheets and turn in only final, clean solutions. Cross out anything you do not want to be considered for grading.

1 Suppose every point of the plane has been colored either red, or blue. Prove that there are two points of the same color whose mutual distance is exactly 1.

2 Suppose $x_1, x_2, \dots, x_{2001}$ are positive real numbers satisfying $\sqrt{x_1} + \sqrt{2x_2} + \cdots + \sqrt{2001 x_{2001}} = 1$. What is the smallest possible value of the sum $x_1 + x_2 + \cdots x_{2001}$?

3 Let $a_n = [(\sqrt{2}+1)^n]$, where $[x]$ denotes the greatest integer $\le x$. Prove that a_n is even if and only if n is odd.

4 Let R_n denote the set of points $(x_1, \dots, x_n)$ in n-dimensional space satisfying $0 \le x_1 \le x_2 \le \cdots \le x_n \le 1$. Evaluate the n-dimensional integral $I_n = \int \dots \int_{R_n} x_1^n x_2^n \dots x_n^n dV$.

5 Evaluate the series

$$\frac{1}{2^1+1} + \frac{2}{2^2+1} + \frac{4}{2^4+1} + \frac{8}{2^8+1} + \cdots$$

6 Let $P_1, P_2, \dots, P_n$ be equally spaced points on the unit circle (so that the points form the vertices of a regular n-gon), and let $S(n)$ denote the sum of the squares of the $\binom{n}{2}$ mutual distances of these n points. Evaluate $S(n)$.