UIUC Department of Mathematics

Solutions to Mock Putnam Exam 4

November 10, 1997

1. What is the largest possible value of the product of positive integers $a_1, a_2, \ldots, a_n$, given that their sum is 1997?

Solution. Suppose $a_1, \ldots, a_n$ are positive integers with sum 1997 and maximal product. First note that none of the integers can be equal to 1, since otherwise replacing 1 and some other number among the a_i by the sum of these two numbers would increase the product while leaving the sum unchanged. Next, if some a_i is greater than 4, then replacing a_i by the two numbers 2 and $a_i - 2$ does not change the sum, but increases the product since $2(x-2) > x$ for $x > 4$. Thus, none of the a_i can be larger than 4. Moreover, since replacing 4 by two 2's leaves the sum and the product unchanged, we may assume without loss of generality that there are no 4's among the a_i. Thus, each a_i is either 2 or 3. If there are more than two 2's among the a_i, then replacing three 2's by two 3's increases the product, but leaves the sum unchanged. Thus, there can be at most two 2's. Since $1997 = 3(665) + 2$, the only way of writing 1997 as a sum of 3's and at most two 2's is by using 665 3's and one 2. The product of these numbers, $2 \cdot 3^{665}$, is the maximal product sought.

2. In a convex n-gon ($n \geq 4$) all diagonals are drawn. How many intersection points in the interior of the n-gon (not counting vertices) are there if no three diagonals intersect at the same point?

Solution. Note that any set of 4 vertices determines exactly one intersection point of two diagonals. Conversely, since no three diagonals intersect at the same point, every intersection point of diagonals is determined in this manner by exactly one subset of 4 vertices (namely the endpoints of the two diagonals). Thus, the number of intersection points is equal to the number of subsets of 4 vertices from the n vertices, i.e., $\binom{n}{4}$.

3. Let $a_1, a_2, \ldots, a_n$ be real numbers with $\sum_{i=1}^n a_i = 1$. Prove that

$$\sum_{i=1}^n i a_i^2 > \frac{1}{2\sqrt{n}}.$$

Solution. By the Cauchy-Schwarz inequality,

$$\begin{aligned} 1 = \left(\sum_{i=1}^n a_i\right)^2 &= \left(\sum_{i=1}^n (a_i i^{1/4})(i^{-1/4})\right)^2 \\ &\leq \left(\sum_{i=1}^n a_i^2 i^{1/2}\right)\left(\sum_{i=1}^n i^{-1/2}\right) < \left(\sum_{i=1}^n a_i^2 i \int_0^n x^{-1/2}\,dx\right) = 2\sqrt{n}\sum_{i=1}^n a_i^2 i. \end{aligned}$$

4. (Putnam '95) Evaluate

$$\sqrt[8]{2207-\cfrac{1}{2207-\cfrac{1}{2207-\cdots}}}$$

Solution. (The second part of this argument is due to Jeff Callahan.) First we show that the continued fraction under the root symbol converges, in the sense that the "truncated fractions" defined by $a_1 = 2207$ and (1) $a_n = 2207 - 1/a_{n-1}$ for $n \geq 2$, converge to a finite limit. From (1) it follows by induction that each a_n is in the interval $[2206, 2207]$. Moreover, since $a_1 > a_2$ and since (1) implies that $a_{n+1} > a_n$ whenever $a_n > a_{n-1}$, it follows that the sequence $\{a_n\}$ is monotonically decreasing and bounded from below by 0. Hence the limit $a = \lim_{n\to\infty} a_n$ exists, and letting $n \to \infty$ on each side of (1) shows that a satisfies the quadratic equation $a^2 - 2207a + 1 = 0$.

Let x be the number sought in the problem. Then $x^8 = a$, so x is a solution to $x^{16} + 1 = 2207x^8$. Adding $2x^8$ to both sides of this equation gives

$$(x^8 + 1)^2 = x^{16} + 2x^8 + 1 = 2209x^8 = (47x^4)^2,$$

which implies $x^8 + 1 = 47x^4$. Adding $2x^4$ to the latter equation gives

$$(x^4 + 1)^2 = (7x^2)^2,$$

which implies $x^4 + 1 = 7x^2$. Finally, adding $2x^2$ to both sides yields

$$(x^2 + 1)^2 = (3x)^2.$$

Since x is positive, this implies $x^2 + 1 = 3x$, from which it follows that $x = (3 + \sqrt{5})/2$ or $x = (3 - \sqrt{5})/2$. However, the latter case is impossible, since then x and therefore $a = x^8$ would be less than 1, contradicting the fact noted above that a is the limit of a sequence of numbers each of which lies in the interval $[2206, 2207]$.

5. Prove that in any party there exist two people that have the same number of friends present. (Assume that friendship is a symmetric relation: If A is among the friends of B, then B is among the friends of A.)

Solution. Let $a_1, a_2, \ldots, a_n$ denote the numbers of friends for the n people present at the party. Then each a_i is a number in the range $\{0, 1, \ldots, n-1\}$. However, if 0 is among the a_i then $n - 1$ is not, since otherwise there would be one person with no friends and another person who is friends with everybody, which is impossible by the assumption that friendship is a reciprocal relation. Thus, there are at most $n - 1$ possible values for the n numbers a_i. By the pigeonhole principle this implies that two of the a_i must be equal.