Mock Putnam Exam 4

23 November 1998

Show all work. Clarity is essential. No calculators allowed.

1. For each positive integer n let P_n and Q_n be polynomials satisfying

$$(x+1)^n = P_n(x^2) - xQ_n(x^2).$$

   Find a simple formula for $(P_n(x))^2 - x(Q_n(x))^2$.

2. Prove that none of the numbers 10001, 100010001, 1000100010001, ... is a prime. (You may assume that the first number in this sequence, 10001, is not a prime.

3. Prove that for every odd integer n the sum $1^n + 2^n + \cdots + n^n$ is divisible by n^2.

4. Let n be a positive integer, and let S be a set of integers in $[0, 2^n)$ such that the binary representations of any two of these integers differ in at least 3 positions. For example, if $n = 4$, then 4 and 9, but not 4 and 8 can both be in the set, since the binary representations of 4 and 9, 0100 and 1001, differ in 3 positions, but not those of 4 and 8. Show that S can contain no more than $2^n/(n+1)$ integers.

5. Transportania is a country with finitely many cities, each of which is directly connected by a road with exactly three other cities. Thus, a traveler who arrives at a city along one of the three roads leading into it can choose between the two other roads, one to his left and one to his right, to continue his trip, assuming that he does not want to return to the city he just came from. Suppose that a traveler starts at city A, goes to city B, there takes the road to his right to city C, then takes the road to his left to city D, and so on, alternating between the left and the right road. Prove that he eventually gets back to city A.