

# Mock Putnam Exam 4

23 November 1998

## Solutions

1. For each positive integer  $n$  let  $P_n$  and  $Q_n$  be polynomials satisfying

$$(1) \quad (x+1)^n = P_n(x^2) - xQ_n(x^2).$$

Find a simple formula for  $(P_n(x))^2 - x(Q_n(x))^2$ .

**Sol:** The formula is (2)  $(P_n(x))^2 - x(Q_n(x))^2 = (1-x)^n$ . Since both sides of (2) are polynomials, it suffices to prove (2) in the case when  $x$  is positive. Setting  $x = y^2$  and applying (1) with  $x = y$  and  $x = -y$ , the left side of (2) becomes

$$\begin{aligned} P_n(y^2)^2 - y^2 Q_n(y^2)^2 &= (P_n(y^2) - yQ_n(y^2))(P_n(y^2) + yQ_n(y^2)) \\ &= (y+1)^n (-y+1)^n = (1-y^2)^n \end{aligned}$$

which is the right-hand side of (1) with  $x = y^2$ .

2. Prove that none of the numbers 10001, 100010001, 1000100010001, ... is a prime. (You may assume that the first number in this sequence, 10001, is not a prime.)

**Sol:** Let  $a_n$  denote the  $n$ -th term in this series. The numbers are of the form  $10^4 + 1$ ,  $10^8 + 10^4 + 1$ , ..., and in general  $a_n = 10^{4n} + 10^{4(n-1)} + \dots + 1$ . Summing this geometric series shows that  $a_n = (10^{4(n+1)} - 1)(10^4 - 1)^{-1}$ . When  $n$  is even, we can factor  $a_n$  into

$$a_n = \frac{10^{2n+2} - 1}{10^2 - 1} \cdot \frac{10^{2n+2} + 1}{10^2 + 1} = \left( \sum_{k=0}^n (100)^k \right) \left( \sum_{k=0}^n (-100)^k \right).$$

Since each of the factors in the last expression is an integer greater than 1, this number must be composite. When  $n$  is odd and greater than 1, say  $n = 2m + 1$ , the factorization

$$a_n = \frac{10^{4m+4} - 1}{10^4 - 1} \cdot (10^{4m+4} + 1) = \left( \sum_{k=0}^m 10^{4k} \right) (10^{4m+4} + 1)$$

shows that  $a_n$  is composite.

3. Prove that for every odd integer  $n$  the sum  $1^n + 2^n + \dots + n^n$  is divisible by  $n^2$ .

**Sol:** For  $n = 1$ , the assertion is trivially true. If  $n$  is odd and greater than 1, we have

$$\begin{aligned} 1^n + 2^n + \dots + n^n &= \sum_{k=1}^{(n-1)/2} (k^n + (n-k)^n) + n^n \\ &= \sum_{k=1}^{(n-1)/2} \left( k^n + n^n + \binom{n}{1} n^{n-1} (-k)^1 + \dots + \binom{n}{n-1} n^1 (-k)^{n-1} + (-k)^n \right) + n^n. \end{aligned}$$

Since the terms  $k^n$  and  $(-k)^n$  cancel and each of the remaining terms is divisible by  $n^2$ , the assertion follows.

4. Let  $n$  be a positive integer, and let  $S$  be a set of integers in  $[0, 2^n)$  such that the binary representations of any two of these integers differ in at least 3 positions. For example, if  $n = 4$ , then 4 and 9, but not 4 and 8 can both be in the set, since the binary representations of 4 and 9, 0100 and 1001, differ in 3 positions, but not those of 4 and 8. Show that  $S$  can contain no more than  $2^n/(n+1)$  integers.

**Sol:** Each element in  $S$  can be represented in binary as a string of  $n$  0's and 1's. Define a "neighbor" of such a string  $s$  as any string that differs from  $s$  in at most one position. (For example, 0100 and 1000 are neighbors.) Obviously, each element  $s \in S$  has exactly  $n+1$  neighbors (including  $s$  itself). Moreover, no  $n$ -digit string can be a neighbor to two distinct elements  $s_1, s_2$  of  $S$  for otherwise  $s_1$  and  $s_2$  could only differ in at most two positions, contrary to the given hypothesis. Thus, the number of 0-1 strings of length  $n$  that are neighbors of some element of  $S$  is at least  $|S|(n+1)$ , where  $|S|$  is the number of elements of  $S$ . On the other hand, this number is at most  $2^n$ . Hence  $|S| \leq 2^n/(n+1)$ .

5. Transportania is a country with finitely many cities, each of which is directly connected by a road with exactly three other cities. Thus, a traveler who arrives at a city along one of the three roads leading into it can choose between the two other roads, one to his left and one to his right, to continue his trip, assuming that he does not want to return to the city he just came from. Suppose that a traveler starts at city  $A$ , goes to city  $B$ , there takes the road to his right to city  $C$ , then takes the road to his left to city  $D$ , and so on, alternating between the left and the right road. Prove that he eventually gets back to city  $A$ .

**Sol:** Suppose each road is a 4-lane road with 2 lanes in each direction, and suppose the traveler takes the left lane if the road he has taken was his "left" choice, and the right lane if it was his "right" choice. The key fact now is that, given the rule of alternating left and right turns, a particular lane determines the itinerary completely forwards and backwards. Since there are finitely many cities, the number of roads, and hence the number of lanes, is finite, so at some point the traveler must hit the same lane twice. Since this lane determines the complete itinerary in both directions, his entire itinerary must be a closed loop and in particular cannot contain any "feeders." Hence he must revisit any place that he has visited before.