UIUC Department of Mathematics

Mock Putnam Exam 4 Solutions

November 15, 1999

Problem 1. Let z_1, z_2, z_3 be complex numbers satisfying (1) $z_1 z_2 z_3 = 1$ and (2) $z_1 + z_2 + z_3 = z_1^{-1} + z_2^{-1} + z_3^{-1}$. Prove that at least one of these numbers is 1.

Solution.
Letting $P = (1 - z_1)(1 - z_2)(1 - z_3)$, the assertion of the problem is equivalent to $P = 0$. However, this follows easily by expanding the product P and using (1) and (2).

Problem 2. Let S be a given sphere with center O and radius r. Let P be any point outside the sphere S, and let S' be the sphere with center P and radius PO. Let A denote the area of the surface of the part of S' that lies inside S. Prove that A is independent of the particular point P chosen.

Solution.
Let Q be a point on the intersection of the two spheres, let α be the angle between the segments PQ and PO, and let $R = |OP| = |PQ|$ be the radius of S'. If h denotes the height of the triangle OPQ, then the surface in question can be obtained by revolving the curve $y = f(x) = \sqrt{R^2 - x^2}$, $0 \le x \le h$, about the y-axis. By the formula for the area of a surface of revolution (learned in 2nd semester calculus), this area is

$$A = \int_0^h 2\pi x \sqrt{1 + f'(x)^2} dx = 2\pi \int_0^h \frac{xR}{\sqrt{R^2 - x^2}} dx = 2\pi R \left(R - \sqrt{R^2 - h^2} \right).$$

Now, by trigonometry, $h = R \sin\alpha$, and $R \sin(\alpha/2) = r/2$, so

$$A = 2\pi R^2 \left(1 - \sqrt{1 - \sin^2 \alpha} \right) = 2\pi R^2 (1 - \cos\alpha) = 2\pi R^2 (2 \sin^2(\alpha/2)) = \pi r^2,$$

which is independent of P, as claimed.

Problem 3. Evaluate

$$\lim_{N\to\infty} \left(\frac{1}{2N+1} + \frac{1}{2N+4} + \frac{1}{2N+7} + \ldots + \frac{1}{2N+3N+1} \right).$$

Solution.
Let S_N denote the expression in parentheses. Then $S_N = R_N/N + 1/(2N+3N+1)$, where $R_N = \sum_{n=0}^{N-1} f(x_n)$ with $f(x) = 1/x$ and $x_n = 2 + (3n+1)/N$. Since the N points x_n are spaced $3/N$ apart and fall into the interval $[2, 5]$, $(3/N)R_N$ is a Riemann sum of "mesh" $3/N$ for the integral $I = \int_2^5 f(x)dx = \log(5/2)$ and therefore converges to this integral as $N \to \infty$. Hence

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} (1/3)(3/N)R_N = 3\log(5/2).$$

Problem 4. Show that for each positive integer n, $2n+1$ divides the binomial coefficient $\binom{2n+1}{n}$.

Solution.
We have

$$\binom{2n}{n} = \frac{(n+1)\binom{2n+1}{n}}{2n+1},$$

Since the left-hand side is an integer and $2n+1$ and $n+1$ have no common prime factors, $2n+1$ must be a divisor of $\binom{2n+1}{n}$.

Problem 5. Find, with proof, the largest real number d such that, for any partition of a square of unit side into two sets, one of the sets contains two points with distance at least d.

Solution.
We claim that the maximal such d is $d = \sqrt{5}/2$. Splitting the square into two $1 \times (1/2)$ rectangles, gives a partition in which the maximimal distance between two points in the same set is $\sqrt{5}/2$. Thus the maximal d with the stated properties cannot be larger $\sqrt{5}/2$, and it remains to show that for any partition of the square into two sets, two points with distance $\geq \sqrt{5}/2$ can be found. Consider the 4 vertices of the square, along with the 4 midpoints of the sides of the square. The distance between two diagonally opposite vertices is $\sqrt{2}$ which is greater than $\sqrt{5}/2$, whereas the distance between a midpoint of one side and a vertex of the opposite side is exactly $\sqrt{5}/2$. It is easy to see that, for any partition of these 8 points (4 vertices and 4 midpoints) into two sets, one of the sets must contain either a pair of diagonally opposite vertices or a midpoint of one side and a vertex on the opposite side. Hence, in any case, there are two points with distance at least $\sqrt{5}/2$.