# UIUC Department of Mathematics

## Mock Putnam Exam 5

## November 24, 1997

1. The functions $f(x) = 4x - 4x^2$ and $\sin \pi x$ agree at $x = 0$, $1/2$, and $1$. Show that $f(x) \geq \sin \pi x$ for $0 \leq x \leq 1$.

2. Let $a_1 = 1$ and $a_{n+1} = 1 + 1/a_n$ for $n \geq 1$. Show that the sequence $a_n$ converges and evaluate its limit.

3. How many sequences of 0's and 1's of length $n$ are there which do not contain blocks of 0's or 1's of length greater than 2? (For example, for $n = 6$ the sequence 011001 would be counted, but not 011100.) Express the answer in terms of a famous sequence.

4. Let $p_n$ $(n = 1, 2, \ldots)$ be a bounded sequence of positive integers that satisfies

$$p_n = \frac{p_{n-1997}p_{n-1} + p_{n-1996}p_{n-2} + \cdots + p_{n-999}p_{n-999}}{p_{n-1}^2 + p_{n-2}^2 + \cdots + p_{n-999}^2} \quad (n \geq 1998).$$

Show that the sequence eventually becomes periodic.

5. Express the infinite series

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

as a rational function of $x$ for $0 < x < 1$.