UIUC Department of Mathematics

Solutions to Mock Putnam Exam 5

November 24, 1997

1. The functions $f(x) = 4x - 4x^2$ and $\sin \pi x$ agree at $x = 0$, $1/2$, and 1. Show that $f(x) \geq \sin \pi x$ for $0 \leq x \leq 1$.

Solution. Since $f(x)$ and $\sin \pi x$ are symmetric about $x = 1/2$, it suffices to prove the asserted inequality for $0 \leq x \leq 1/2$. Let $g(x) = f(x) - \sin \pi x$. We have $g'(x) = 4 - 8x - \pi \cos \pi x$ and $g''(x) = -8 + \pi^2 \sin \pi x$. Thus $g''(x)$ increases monotonically from -8 to $\pi^2 - 8 > 0$ as x ranges over the interval $[0, 1/2]$ and therefore has a unique zero x_0 in this interval. It follows that $g'(x)$ is increasing for $0 \leq x < x_0$ and decreasing for $x_0 < x \leq 1$. Since $g(0) = 0$ and $g'(0) = 4 - \pi > 0$, this implies that $g(x) \geq 0$ in the interval $[0, x_0]$ and $g(x_0) > 0$. In the interval $[x_0, 1]$, $g(x)$ is concave downwards, and since $g(x_0) > 0$ and $g(1) = 0$ it follows that $g(x) \geq 0$ in that interval.

2. Let $a_1 = 1$ and $a_{n+1} = 1 + 1/a_n$ for $n \geq 1$. Show that the sequence a_n converges and evaluate its limit.

Solution. First, since $a_1 = 1$, the recurrence (1) $a_{n+1} = 1 + 1/a_n$ implies by induction that $a_n \geq 1$ for all $n \geq 1$. Feeding the inequality $a_n \geq 1$ back into (1) shows that $a_n \leq 2$ for all $n \geq 1$. Using this bound in (1) again, we obtain (2) $3/2 \leq a_n \leq 2$ for all $n \geq 2$.

To show the convergence of the sequence $\{a_n\}$, set $d_n = a_{n+1} - a_n$. and note that by (1) we have

$$d_n = a_{n+1} - a_n = \frac{1}{a_n} - \frac{1}{a_{n-1}} = \frac{-d_{n-1}}{a_n a_{n-1}}.$$

for $n \geq 2$. By (2) it follows that $|d_n| \leq (2/3)|d_{n-1}|$ for $n \geq 2$ which by induction implies $|d_n| \leq (2/3)^{n-1}|d_1|$. Hence the series $\sum_{n=1}^{\infty} d_n$ converges, and since $a_n = a_1 + \sum_{k=1}^{n-1} d_k$, this proves the convergence of the sequence $\{a_n\}$. To evaluate the limit $\alpha = \lim_{n\to\infty} a_n$, let $n \to \infty$ on each side of (1). Then $\alpha = 1 + 1/\alpha$, or $\alpha^2 - \alpha - 1 = 0$. Solving this quadratic equation gives $\alpha = (1 \pm \sqrt{5})/2$, and since, by (2), $3/2 \leq \alpha \leq 2$, it follows that $\alpha = (1 + \sqrt{5})/2$.

3. How many sequences of 0's and 1's of length n are there which do not contain blocks of 0's or 1's of length greater than 2? (For example, for $n = 6$ the sequence 011001 would be counted, but not 011100.) Express the answer in terms of a famous sequence.

Solution. Let a_n be the number of such sequences. Clearly $a_1 = 2$ (the admissible sequences of length 1 are 0 and 1) and $a_2 = 4$ (with 00, 01, 10, 11 as admissible sequences).

Suppose now $n \geq 3$. Let b_n denote the number of admissible sequences of length n ending in 1 and c_n the number of such sequences ending in 0. By symmetry, $(*)$ $b_n = c_n = a_n/2$. The sequences counted by b_n fall into two classes, those ending in the string 01 and those ending in the string 011. The number of sequences in the first class is exactly equal to the number of admissible sequences of lenght $n-1$ ending in 0, i.e., c_{n-1}. Similarly, the number of sequences in the second class is equal to c_{n-2}. Thus, we have $b_n = c_{n-1} + c_{n-2}$ for $n \geq 3$, and by $(*)$ we deduce that $a_n = a_{n-1} + a_{n-2}$. This shows that the numbers a_n satisfy the same recurrence as the Fibonnacci numbers F_n defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Since $a_1 = 2 = 2F_2$ and $a_2 = 4 = 2F_3$, it follows that $a_n = 2F_{n+1}$ for all $n \geq 1$.

4. Let p_n $(n = 1, 2, \ldots)$ be a bounded sequence of positive integers that satisfies

$$p_n = \frac{p_{n-1997}p_{n-1} + p_{n-1996}p_{n-2} + \cdots + p_{n-999}p_{n-999}}{p_{n-1}^2 + p_{n-2}^2 + \cdots + p_{n-999}^2} \qquad (n \geq 1997).$$

Show that the sequence eventually becomes periodic.

Solution. Let $T(n)$ denote the 1997-tuple $(p_{n-1}, \ldots, p_{n-1997})$. By the given recurrence for $p(n)$, the value of $p(n)$ is completely determined by the value of the tuple $T(n)$. Since the numbers $p(n)$ are bounded positive integers, there are only finitely many possible values for $T(n)$ (namely at most M^{1997} where M is such that $1 \leq p(n) \leq M$ for all n). By the pigeon hole principle, it follows that there exist positive integers $n_1 < n_2$ with $T(n_1) = T(n_2)$. In view of the above remark, this implies $p(n_1) = p(n_2)$, which in turn implies $T(n_1 + 1) = T(n_2 + 1)$ and therefore $p(n_1 + 1) = p(n_2 + 1)$. By induction we conclude that $T(n_1 + k) = T(n_2 + k)$ and $p(n_1 + k) = p(n_2 + k)$ for all positive integers k. Thus, $p(n)$ is eventually periodic with period $n_2 - n_1$.

5. Express the infinite series

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

as a rational function of x for $0 < x < 1$.

Solution. [The condition $0 < x < 1$ was missing in the original version of the exam. As Jeff Callahan observed, the given series also converges when $x > 1$, but to a different rational function (namely $1/(1-x)$).]

Expanding $(1 - x^{2^{n+1}})^{-1}$ into a geometric series, we see that the series equals

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^{2^n + m2^{m+1}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^{2^n(1+2m)}.$$

In the last sum each positive integers occurs exactly once as an exponent since each positive integer has a unique representation of the form $2^n k$ where $n \geq 0$ and k is an odd positive integer. Thus, the given sum equals $\sum_{n=1}^{\infty} x^n = x/(1-x)$ which is the rational function sought in the problem. The rearranging of the terms in the above double sum is justfied by the absolute convergence of this double series (or, equivalently, the series $\sum_{n=1}^{\infty} x^n$) when $|x| < 1$.