

UIUC Department of Mathematics

Mock Putnam Exam 5

January 30, 1999

Solutions will be posted within a few days at
<http://www.math.uiuc.edu/~hildebr/putnam/mockputnam.html>.

1. Given 10 positive integers a_1, a_2, \dots, a_{10} , show that there exist numbers $\epsilon_i \in \{+1, 0, -1\}$ ($i = 1, 2, \dots, 10$), not all 0, such that the sum $\sum_{i=1}^{10} \epsilon_i a_i$ is divisible by 1000.
2. Let f be a function from the positive integers into the positive integers and satisfying $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n . Find $f(100)$.
3. [Putnam 82, A3] Evaluate the integral

$$\int_0^{\infty} \frac{\arctan(\pi x) - \arctan x}{x} dx.$$

4. [Putnam 71, B1] Let S be a set and $*$ a binary operation on S satisfying $x * x = x$ for all $x \in S$ and $(x * y) * z = (y * z) * x$ for all $x, y, z \in S$. Prove that $*$ is commutative (i.e., $x * y = y * x$ for all $x, y \in S$).
5. Let x be a real number between 0 and 1. Evaluate the sum $\sum_{n=1}^{\infty} (-1)^{[2^n x]} 2^{-n}$, where $[t]$ denotes the greatest integer less than or equal to t .