UIUC Department of Mathematics

Mock Putnam Exam 5

January 30, 1999

Solutions

1. Given 10 positive integers $a_1, a_2, \ldots, a_{10}$, show that there exist numbers $\epsilon_i \in \{+1, 0, -1\}$ ($i = 1, 2, \ldots, 10$), not all 0, such that the sum $\sum_{i=1}^{10} \epsilon_i a_i$ is divisible by 1000.

Solution: Consider the sums $S(I) = \sum_{i \in I} a_i$, where $I \subset \{1, 2, \ldots, 10\}$. There are $2^{10} = 1024$ such sums, so by the pigeonhole principle two of these sums, say $S(I_1)$ and $S(I_2)$ (with $I_1 \neq I_2$) must have the same remainder when divided by 1000. The difference $S_{I_1} - S_{I_2}$ is then divisible by 1000 and can be written in the form $\sum_{i=1}^{10} \epsilon_i a_i$ with $\epsilon_i = 1$ if $i \in I_1 \setminus I_2$, $\epsilon_i = -1$ if $i \in I_2 \setminus I_1$, and $\epsilon_i = 0$ otherwise. Since $I_1 \neq I_2$, not all ϵ_i are 0.

2. Let f be a function from the positive integers into the positive integers and satisfying $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n. Find $f(100)$.

Solution: We will show that, for any integers $k \geq 0$ and $0 \leq m < 3^k$, $(*)$ $f(3^k + m) = 2 \cdot 3^k + m$. Since $100 = 3^4 + 19$, we obtain from $(*)$ $f(100) = f(3^4 + 19) = 2 \cdot 3^4 + 19 = 181$.

Proof of $(*)$: We first observe that the first condition on f implies (1) $f(n+m) \geq f(n) + m$ for any positive integers n and m. Next, let $a = f(1)$. If $a > 3$ then the second condition implies $f(a) = 3$ which contradicts (1) with $n = a - 1$ and $m = 1$. If $a = 1$ then we get $3 = f(f(1)) = f(a) = f(1) = a$ which is a contradiction. Finally, if $a = 3$, then we have $3 = f(f(1)) = f(3)$, whereas (1) implies $f(3) \geq f(1) + 2 = a + 2 = 5$, so we get again a contradiction. Thus, we necessarily have (2) $f(1) = 2$, which in turn implies (3) $f(2) = f(f(1)) = 3$. We now use induction to show that, for any nonnegative integer k, (4) $f(2 \cdot 3^k) = 3^{k+1}$ and (5) $f(3^k) = 2 \cdot 3^k$.

For $k = 0$, (4) and (5) reduce to (2) and (3). Assume now that (4) and (5) both hold for some nonnegative integer k. Then $f(3^{k+1}) = f(f(2 \cdot 3^k)) = 3 \cdot 2 \cdot 3^k = 2 \cdot 3^{k+1}$ and $f(2 \cdot 3^{k+1}) = f(f(3^{k+1})) = 3 \cdot 3^{k+1} = 3^{k+2}$, which proves these formulas for $k + 1$ and thus completes the induction.

From (1), (4) and (5), we see that the values $f(3^k + m)$, $m = 0, 1, \ldots 3^k - 1$ must form an increasing sequence of 3^k distinct integers, all contained in the interval $[2 \cdot 3^k, 3 \cdot 3^k - 1]$. Since there are exactly 3^k integers in that interval, these values must fill the entire interval, i.e., we have $f(3^k+m) = 3^k+m$ for $0 \le m < 3^k$. This proves $(*)$

3. [Putnam 82, A3] Evaluate the integral

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan x}{x} dx.$$

Solution: The given integral is

$$\begin{aligned} &\lim_{t\to\infty} \left(\int_0^t \frac{\arctan(\pi x)}{x} dx - \int_0^t \frac{\arctan(x)}{x} dx \right) \\ &= \lim_{t\to\infty} \int_t^{\pi t} \frac{\arctan(x)}{x} dx = \lim_{t\to\infty} \int_t^{\pi t} \frac{\pi/2}{x} dx = \frac{\pi}{2} \ln \pi, \end{aligned}$$

since $\arctan(x) \to \pi/2$ as $x \to \infty$.

4. [Putnam 71, B1] Let S be a set and $*$ a binary operation on S satisfying $x * x = x$ for all $x \in S$ and $(x * y) * z = (y * z) * x$ for all $x, y, z \in S$. Prove that $*$ is commutative (i.e., $x * y = y * x$ for all $x, y \in S$).

Solution: Applying the two given rules repeatedly, we have, for any $x, y \in S$,

$$\begin{aligned} x * y &= (x * y) * (x * y) = ((x * y) * x) * y = ((y * x) * x) * y = ((x * x) * y) * y \\ &= (x * y) * y = (y * y) * x = y * x, \end{aligned}$$

which proves the commutativity of $*$.

5. Let x be a real number between 0 and 1. Evaluate the sum $\sum_{n=1}^\infty (-1)^{[2^n x]} 2^{-n}$, where $[t]$ denotes the greatest integer less than or equal to t.

Solution: The binary expansion of x can be written as $x = \sum_{i=1}^\infty a_i 2^{-i}$ where each a_i is 0 or 1 and the sequence $\{a_i\}$ does not end in all 1's. Then

$$2^n x = 2^{n-1} a_1 + 2^{n-2} a_2 + \ldots + a_n + a_{n+1} 2^{-1} + a_{n+2} 2^{-2} + \ldots,$$

so $[2^n x] = 2b_n + a_n$ for some integer b_n. It follows that $(-1)^{[2^n x]} = (-1)^{2b_n + a_n} = 1 - 2a_n$. The given sum therefore becomes $\sum_{n=1}^\infty (1 - 2a_n) 2^{-n} = 1 - 2x$.