# UIUC Department of Mathematics

# Mock Putnam Exam 6 Solutions

# February 2, 1998

1. Find an explicit formula for the $n$-th term of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, \ldots$$

**Solution.** Let $f(n)$ denote the $n$th term in the given sequence, and let $g(k)$ denote the first index $n$ for which $f(n) = k$. Since the sequence is non-decreasing and, for each $k = 1, 2, \ldots$, the value $k$ appears exactly $k$ times in the sequence, we have $g(k) = 1 + \sum_{i=1}^{k-1} i = 1 + k(k-1)/2$. Moreover, we have $f(n) = k$ if and only if $g(k) \le n < g(k+1)$, i.e., if

$$\frac{1}{2}k^2 - \frac{1}{2}k + 1 \le n < \frac{1}{2}k^2 + \frac{1}{2}k + 1.$$

The latter relation is equivalent to

$$\left(k - \frac{1}{2}\right)^2 \le 2n - \frac{7}{4} < \left(k + \frac{1}{2}\right)^2,$$

which in turn holds if and only if

$$\sqrt{2n - 7/4} - 1/2 < k \le \sqrt{2n - 7/4} + 1/2.$$

Hence $f(n) = \lfloor \sqrt{2n - 7/4} + 1/2 \rfloor$.

2. Determine (by an explicit formula) the number of subsets of an $n$-element set that have an *even* number of elements.

**Solution.** The number in question is

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = \frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} (1 + (-1)^m) = \frac{1}{2} ((1+1)^n + (1-1)^n) = 2^{n-1},$$

by the binomial theorem. (Thus, exactly half of the $2^n$ subsets of an $n$-element set have an even number of elements.)
**Remark:** When $n$ is odd, this can also be seen by noting that a subset $A$ of has an even number of elements if and only if its complement $A^c$ has an odd number of elements. Thus, there is a $1-1$ correspondence between subsets with an even number of elements and those

with an odd number of elements, and so there are as many subsets with an even number of elements as there are subsets with an odd number of elements.

3. Determine the least value of $x_1^2 + x_2^2 + \cdots + x_{10}^2$, where $x_1, x_2, \ldots, x_{10}$ are real numbers satisfying $(*)$ $\sum_{n=1}^{10} x_n\sqrt{n} = 1$.

**Solution.** Let $S(x_1, \ldots, x_{10}) = \sum_{n=1}^{10} x_n^2$. By Cauchy's inequality, we have for any numbers $x_1, \ldots, x_{10}$ satisfying the given constraint,

$$(1) \qquad 1 = \left(\sum_{n=1}^{10} x_n\sqrt{n}\right)^2 \leq \left(\sum_{n=1}^{10} x_n^2\right)\left(\sum_{n=1}^{10} n\right) = \frac{10\cdot 11}{2} S(x_1, \ldots, x_{10}).$$

Thus, $S(x_1, \ldots, x_{10})$ must be at least $1/55$. On the other hand, with the choice $x_n = (1/55)\sqrt{n}$, the inequality in (1) becomes an equality, and we therefore have $S(x_1, \ldots, x_{10} = 1/55$, while $(*)$ is satisfied. Hence, $1/55$ is the least value of $S(x_1, \ldots, x_{10})$ subject to $(*)$.

4. Evaluate the integral

$$I_n = \int_0^\pi \left(\frac{\sin(nx)}{\sin x}\right)^2 dx$$

for all positive integral values of $n$.

**Solution.** We show that $I_n = n\pi$. Clearly, $I_1 = \pi$, so it suffices to show that, for $n \geq 2$, $I_n - I_{n-1} = \pi$. From the identity

$$\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$$

(which can be derived from the identities for $\sin(\alpha \pm \beta)$ we have $\sin^2 nx - \sin^2(n-1) = \sin(2n-1)x \sin x$. Hence

$$I_n - I_{n-1} = \int_0^\pi \frac{\sin^2 nx - \sin^2(n-1)x}{\sin^2 x} dx = \int_0^\pi \frac{\sin(2n-1)x}{\sin x} dx = J_{2n-1},$$

say, and it suffices to show that for odd values of $m$, $J_m = \pi$. Using the identity

$$\sin\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2} = \frac{1}{2}(\sin\alpha - \sin\beta)$$

with $\alpha = (m-2)x$ and $\beta = mx$, we see that, for $m \geq 2$,

$$J_m - J_{m-2} = \int_0^\pi = 2\int_0^\pi \frac{\sin(mx) - \sin(m-2)x}{\sin x} dx = \int_0^\pi \cos(m-1)x dx = 0.$$

Hence $J_{2n-1} = J_{2n-3} = \cdots = J_1 = 2\int_0^\pi dx = \pi$.

5. A car dealership that was open 7 days a week sold at least one car each day in 1997, and a total of 600 cars during that year. Prove that there was a period of consecutive days during which exactly 129 cars were sold. (Note that there were 365 days in 1997.)

**Solution.** Let $a_n$ denote the number of cars sold in days $1, 2, \ldots, n$ of the year. The given assumptions yield $1 \leq a_1 < a_2 < \cdots < a_{365} = 600$; in particular, the $a_n$ are distinct positive integers in the range $\{1, 2, \ldots, 600\}$. Note that for $0 \leq i < j \leq 365$, $a_j - a_i$ is the number of cars sold on days $i+1, i+2, \ldots, j$. Thus, it suffices to show that $(*)$ $a_i = a_j + 129$ holds for some indices $i$ and $j$. To this end we apply the pigeon hole principle to the list of numbers $a_1, \ldots, a_{365}, a_1 + 129, \ldots, a_{365} + 129$. Since each $a_i$ is an integer in the interval $[1, 600]$, these 730 numbers are integers in the interval $[1, 729]$. By the pigeonhole principle it follows that two of these integers must be the same. Since the integers $a_i$ are distinct (as are the integers $a_i + 129$) this is only possible if, for some indices $i$ and $j$, $a_i = a_j + 129$. This proves $(*)$.