Advanced Putnam Training Session 2: Real analysis, limits, and convergence of series

1. Determine, with proof, the set of all positive real numbers a for which the inequality $a^x \geq x^a$ is true for all positive real numbers x.

   Solution: The given inequality is equivalent to $a^{1/a} \geq x^{1/x}$. The latter inequality holds for *all* $x > 0$, if and only if the function $f(x) = x^{1/x}$ $(x > 0)$ attains its maximum at $x = a$. Now, $f(x)$ is maximal if and only if $g(x) = \ln f(x) = (1/x)\ln x$ is maximal. Now $g'(x) = (1 - \ln x)/x^2$, which is positive for $0 < x < e$ and negative for $x > e$. Thus the function g, and hence also f, has a unique maximum at $a = e$. It follows that the set of numbers a sought consists of the single number $a = e$.

2. (B4, Putnam 1988) Suppose a_n are *positive* real numbers such that the series $\sum_{n=1}^\infty a_n$ converges. Show that the series $\sum_{n=1}^\infty a_n^{n/(n+1)}$ converges as well.

   Solution: Split $\mathbb{N}$, the set of positive integers, into sets A_i, $i = 1, 2, 3$, defined as

$$\begin{aligned} A_1 &= \{n \in \mathbb{N} : a_n > 1\}, \\ A_2 &= \{n \in \mathbb{N} : 2^{-n} < a_n \leq 1\}, \\ A_3 &= \{n \in \mathbb{N} : a_n \leq 2^{-n}\}. \end{aligned}$$

   To prove the convergence of $\sum_{n\in\mathbb{N}} a_n^{n/(n+1)}$, it is enough to show that for each $i = 1, 2, 3$,

$$\sum_{n\in A_i} a_n^{n/(n+1)} < \infty. \qquad (*)$$

   Since $\sum_{n\in\mathbb{N}} a_n$ converges, we have $a_n \to 0$ as $n \to \infty$, so the set A_1 is finite, and hence $(*)$ holds for $i = 1$.

   Next, since for any x with $0 < x \leq 1$, $x^{n/(n+1)} \leq x^{1/2}$, we have

$$\begin{aligned} \sum_{n\in A_3} a_n^{n/(n+1)} &\leq \sum_{n\in A_3} a_n^{1/2} < \sum_{n\in A_3} 2^{-n/2} \\ &\leq \sum_{n=1}^{\infty} 2^{-n/2} < \infty, \end{aligned}$$

   proving $(*)$ for $i = 3$.

   Finally, since for $2^{-n} < x \leq 1$, $x^{n/(n+1)} = x \cdot x^{-1/(n+1)} < x \cdot 2^{n/(n+1)} < 2x$, we have

$$\begin{aligned}\sum_{n\in A_2} a_n^{n/(n+1)} &< \sum_{n\in A_2} 2a_n \\ &\le 2\sum_{n=1}^{\infty} a_n < \infty.\end{aligned}$$

Hence $(*)$ holds for $i = 2$ as well.

3. (U of I Undergraduate Math Contest 2002) Determine, with proof, whether the series

$$\sum_{\substack{m,n=1\\ m<n}}^{\infty} \left(\frac{m}{n}\right)^{mn},$$

converges. (The summation runs over all pairs (m,n) of positive integers with $m < n$.)

Solution: Write the given double sum as $\sum_{n=2}^{\infty} S_n$ with

$$S_n = \sum_{m=1}^{n-1} \left(\frac{m}{n}\right)^{mn}.$$

We now estimate the terms in S_n, considering separately the cases $1 \le m < n/2$ and $n/2 \le m \le n-1$. If $1 \le m \le n/2$, then

$$\left(\frac{m}{n}\right)^{mn} \le \left(\frac{1}{2}\right)^{mn} \le 2^{-n}.$$

If $n/2 < m \le n-1$, then

$$\left(\frac{m}{n}\right)^{mn} \le \left(\frac{n-1}{n}\right)^{mn} \le \left(1-\frac{1}{n}\right)^{n^2/2} \le \left(e^{-(1/n)}\right)^{n^2/2} = e^{-n/2}.$$

Since S_n contains at most n terms, it follows that

$$S_n \le n\min(2^{-n}, e^{-n/2}) = ne^{-n/2}.$$

Hence the given series is bounded by

$$\sum_{n=2}^{\infty} S_n \le \sum_{n=2}^{\infty} ne^{-n/2},$$

and since the latter series converges (e.g., by the root test), it follows that the given series is convergent.

4. (U of I Undergraduate Math Contest 2005) Determine, with proof, whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.7+\sin n}}$$

converges or diverges.

Solution: We show that the series diverges. Note that $\sin x \le -\sqrt{3}/2$ whenever x falls into one of the intervals

$$I_k = [(2k+4/3)\pi, (2k+5/3)\pi], \quad k = 0, \pm 1, \pm 2, \dots$$

Each of these intervals has length $\pi/3 > 1$ and the gap between two successive intervals has length $< (5/3)\pi < 6$. Hence, among any 7 consecutive integers n at least one must fall into one of the intervals I_k; for this value of n we have $1.7 + \sin n < 1.7 - \sqrt{3}/2 < 1.7 - 1.5/2 = 0.95 < 1$, so the corresponding term in the above series is greater than $1/n$. Therefore the above series is bounded from below by

$$\sum_{m=0}^{\infty} \sum_{n=7m+1}^{7m+7} \frac{1}{n^{1.7+\sin n}} \ge \sum_{m=0}^{\infty} \frac{1}{7m+7} = \frac{1}{7} \sum_{m=1}^{\infty} \frac{1}{m} = \infty,$$

and hence diverges.

5. Let $\{c_n\}_{n\ge1}$ be a sequence of positive numbers, and suppose that, for any sequence $\{d_n\}_{n\ge1}$ of positive numbers with $\lim_{n\to\infty} d_n = 0$, the series $\sum_{n=1}^{\infty} c_n d_n$ converges. Show that the series $\sum_{n=1}^{\infty} c_n$ converges as well.

Solution: We argue by contradiction. Given a sequence $\{c_n\}$ of positive numbers such that $\sum_{n=1}^{\infty} c_n$ diverges, we will construct a sequence $\{d_n\}$ of positive numbers tending to 0, such that $\sum_{n=1}^{\infty} c_n d_n$ diverges. This will prove the assertion.

So suppose $c_n > 0$ and $\sum_{n=1}^{\infty} c_n$ diverges. Then the partial sums $\sum_{n=1}^{N} c_n$ tend to infinity as $N \to \infty$. It follows that we can break up the series into "chunks" whose sizes tend to infinity; specifically, there exists a sequence $1 = N_0 < N_1 < \cdots$ of integers such that if $S_i = \sum_{n=N_i}^{N_{i+1}-1} c_n$, then $(*)$ $S_i \to \infty$ as $i \to \infty$. Now define $d_n = 1/S_i$ if $N_i \le n < N_{i+1}$. Then $(*)$ implies that $d_n \to 0$ as $n \to \infty$, so the sequence $\{d_n\}$ satisfies the conditions stated in the problem. On the other hand, by construction we have $\sum_{n=N_i}^{N_{i+1}-1} c_n d_n = 1$ for each i, so $\sum_{n=1}^{\infty} c_n d_n$ diverges. This is the desired contradiction.

6. (A6, Putnam 1987) Let A denote the set of positive integers whose decimal expansion does not have a zero. Determine, with proof, whether the infinite series $\sum_{n\in A} 1/n$ converges.

Solution: We break the range of summation into finite intervals $I_k = 10^{k-1} \le n < 10^k$, $k = 1, 2, \ldots$, and let $S_k = \sum_{n \in I_k \cap A} 1/n$ denote the corresponding partial sum. Now, note that an integer $n \in I_k$ has at most k digits, and if n is also in the set A, then there are only 9 possible values for each of these digits. Hence the total number of elements in $I_k \cap A$ is at most 9^k. On the other hand, if $n \in I_k$ then $n \ge 10^{k-1}$. Thus,

$$S_k = \sum_{n \in I_k \cap A} \frac{1}{n} \le \frac{1}{10^{k-1}} \#\{n : n \in I_k \cap A\} \le \frac{9^k}{10^{k-1}},$$

and therefore

$$\sum_{n \in A} \frac{1}{n} = \sum_{k=1}^{\infty} S_k \le 10 \sum_{k=1}^{\infty} \left(\frac{9}{10}\right)^k = \frac{9}{1 - 9/10} = 90 < \infty.$$

Hence the series converges.

7. (U of I Undergraduate Math Contest 2002) Let $a_1 = \sqrt{2}$, and for $n > 1$ define a_n by $a_n = (\sqrt{2})^{a_{n-1}}$. Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ converges and determine its limit.

Solution: We first show that the sequence is monotone increasing, i.e., that

$$a_{n-1} < a_n \quad (n \ge 2). \tag{1}$$

We use induction on n: The base case, $n = 2$, is clear since $a_1 = \sqrt{2}$ and $a_2 = \sqrt{2}^{\sqrt{2}} = 2^{\sqrt{2}/2} > 2^{1/2} = a_1$. Now let $n \ge 2$ and assume that (1) holds for this value of n. Then

$$a_n = \sqrt{2}^{a_{n-1}} < \sqrt{2}^{a_n} = a_{n+1},$$

since the function $f(x) = \sqrt{2}^x$ is monotone. Hence (1) holds with $n + 1$ in place of n, and the induction is complete.

Next, we show

$$a_n < 2 \quad (n \ge 1). \tag{2}$$

Again we use induction. Clearly, (2) holds for $n = 1$. Now let $n \ge 2$ and assume that (2) holds for this value of n. Then

$$a_{n+1} = \sqrt{2}^{a_n} < \sqrt{2}^2 = 2,$$

so (2) remains valid for $n + 1$, and the induction is complete.

By (1) and (2) the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone increasing and bounded. By a general result in analysis, any such sequence converges. Hence the limit $A = \lim_{n\to\infty} a_n$ exists and is finite. To find A, we use a standard trick: We let $n \to \infty$ on each side of the recurrence $a_n = \sqrt{2}^{a_{n-1}}$, getting the equation

(*) $A = \sqrt{2}^A$, so A must be a solution to (*). Clearly $A = 2$ is a solution to this equation, though not necessarily the only one, so we still need to show that $A = 2$ is indeed the solution we are looking for.

To this end, first note that all terms a_n are positive (since $a_n = \sqrt{2}^{a_{n-1}} > 0$) and that, by (2) all terms are less than 2. It follows that the limit $A = \lim_{n\to\infty} a_n$ satisfies $0 \le A \le 2$. (Note the inequality signs. Even though the a_n's are strictly between 0 and 2, their limit may be one of the boundary values 0 or 2.)

Next note that the function $x - \sqrt{2}^x$ has derivative $1 - \sqrt{2}^x (1/2)\ln 2$ which is $\ge 1 - \ln 2 > 0$ for $x \le 2$. Hence this function is strictly increasing for $x \le 2$ and therefore cannot have more than one zero in this range. Thus, the solution $A = 2$ is the only solution of (*) in the range $A \le 2$, and hence must be equal to $\lim_{n\to\infty} a_n$.

8. (B5, Putnam 1969; U of I Mock Putnam Exam 2009) Let $\mathcal{A}$ be an infinite set of positive integers, and let $A(n)$ denote the number of elements of $\mathcal{A}$ that are $\le n$. Suppose that the series $\sum_{a\in\mathcal{A}} 1/a$ converges. Show that $\lim_{n\to\infty} A(n)/n = 0$.

   Solution: We argue by contradiction. Suppose $A(n)/n$ does not tend to 0. Then there exists $\epsilon > 0$ and an infinite sequence $n_1 < n_2 < \cdots$ of positive integers such that $A(n_k)/n_k \ge \epsilon$ for all k. Let N be a fixed positive integer. Since $n_k \to \infty$ as $k \to \infty$, we have $n_k > N$ for all large enough k. For such k we have

   $$\sum_{\substack{a\in\mathcal{A}\\ N<a\le n_k}} \frac{1}{a} \ge \sum_{\substack{a\in A\\ N<a\le n_k}} \frac{1}{n_k} \ge \frac{A(n_k) - A(N)}{n_k} \ge \epsilon - \frac{A(N)}{n_k}.$$

   Letting $k \to \infty$, we conclude

   $$\sum_{a\in\mathcal{A}, a>N} \frac{1}{a} \ge \epsilon.$$

   Since N was arbitrary, the series $\sum_{a\in\mathcal{A}} 1/a$ cannot be convergent.