Advanced Putnam Training Session 2: Real analysis, limits, and convergence

1. Determine, with proof, the set of all positive real numbers a for which the inequality $a^x \geq x^a$ is true for all positive real numbers x.

2. (B4, Putnam 1988) Suppose a_n are *positive* real numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges. Show that the series $\sum_{n=1}^{\infty} a_n^{n/(n+1)}$ converges as well.

3. (U of I Undergraduate Math Contest 2002) Determine, with proof, whether the series
$$\sum_{\substack{m,n=1 \\ m<n}}^{\infty} \left(\frac{m}{n}\right)^{mn},$$
converges. (The summation runs over all pairs (m,n) of positive integers with $m < n$.)

4. (U of I Undergraduate Math Contest 2005) Determine, with proof, whether the series
$$\sum_{n=1}^{\infty} \frac{1}{n^{1.7+\sin n}}$$
converges or diverges.

5. Let $\{c_n\}_{n\geq 1}$ be a sequence of positive numbers, and suppose that, for any sequence $\{d_n\}_{n\geq 1}$ of positive numbers with $\lim_{n\to\infty} d_n = 0$, the series $\sum_{n=1}^{\infty} c_n d_n$ converges. Show that the series $\sum_{n=1}^{\infty} c_n$ converges as well.

6. (A6, Putnam 1987) Let A denote the set of positive integers whose decimal expansion does not have a zero. Determine, with proof, whether the infinite series $\sum_{n\in A} 1/n$ converges.

7. (U of I Undergraduate Math Contest 2002) Let $a_1 = \sqrt{2}$, and for $n > 1$ define a_n by $a_n = (\sqrt{2})^{a_{n-1}}$. Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ converges and determine its limit.

8. (B5, Putnam 1969; U of I Mock Putnam Exam 2009) Let $\mathcal{A}$ be an infinite set of positive integers, and let $A(n)$ denote the number of elements of $\mathcal{A}$ that are $\leq n$. Suppose that the series $\sum_{a\in\mathcal{A}} 1/a$ converges. Show that $\lim_{n\to\infty} A(n)/n = 0$.