Advanced Putnam Training Session 6: More number theory

1. (B1, Putnam 1988) Show that every composite integer n is expressible as $n = xy + xz + yz + 1$, with x, y, z positive integers.

   Solution: This looks a lot more complicated than it is. If n is composite, then $n = ab$ with $a, b \geq 2$ integers. Letting $x = a - 1$, $y = b - 1$, this becomes $n = (x+1)(y+1) = xy + x + y + 1$, which is of the required form with $z = 1$.

2. For which pairs (n, m) of positive integers is $\sqrt{n} + \sqrt{m}$ rational?

   Solution: Consider $x = \sqrt{n} + \sqrt{m}$, $y = \sqrt{n} - \sqrt{m}$. Then $xy = n - m$ is rational. Hence, x and y must be either both rational or both irrational. If x and y are both rational, then so is $x + y = 2\sqrt{n}$, so n must be a perfect square. (This uses the fact that, for any positive integer a, $\sqrt{a}$ is rational if and only if $a = b^2$ for some positive integer b. In other words, $\sqrt{a}$ is either an integer, or irrational.) Arguing similarly with $x - y = 2\sqrt{m}$ shows that m must also be a perfect square. Conversely, if n and m are perfect squares, then $\sqrt{n}$ and $\sqrt{m}$ are integers, so x and y are rational. Thus, $x = \sqrt{n} + \sqrt{m}$ is rational if and only if m and n are perfect squares.

3. For which pairs (m, n) of integers ≥ 2 is $\log_m n$ rational?

   Solution: Suppose $\log_m n = r/s$, where r/s is a rational in reduced form, i.e., with positive integers r and s that are relatively prime. Then $n = m^{r/s}$ or $(*)$ $n^s = m^r$. For any prime p let p^{α_p} be the power of p that divides m, and p^{β_p} the power of p that divides n. By unique prime factorization, the exponents of p on both sides of $(*)$ must be equal, i.e., $s\beta_p = r\alpha_p$. Since we assumed r and s to be relative prime, this implies that r divides β_p for every p. Hence $n_0 = n^{1/r} = \prod_p p^{\beta_p/r}$ is an integer. Similarly, $m_0 = m^{1/s}$ is an integer. Moreover, $(*)$ implies $n_0 = n^{1/r} = (n^s)^{1/rs} = (m^r)^{1/rs} = m_0$. Therefore (n, m) must be of the form $(**)$ (n_0^r, n_0^s), where n_0 is an integer ≥ 2, and r and s are positive integers. Conversely, if (m, n) is of the form $(**)$, then $\log_m n = r/s$ is rational. Thus, the requested set of pairs (m, n) is exactly the set of integers of the form $(**)$ with $n_0 \geq 2$ and $r, s \geq 1$.

4. Show that all integers of the form $n^4 + 4^n$, $n = 2, 3, \ldots$ are composite.

   Solution: This is a very well-known problem. The solution hinges on the identity
   $$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab),$$
   which shows that any number of the form $a^4 + 4b^4$ with a, b positive integers such that $(*)$ $a^2 + 2b^2 - 2ab > 1$ is composite. Now $(*)$ is equivalent to $(a - b)^2 + b^2 > 1$, so $(*)$ holds for all pairs (a, b) of positive integers except $(a, b) = (1, 1)$.

To apply this to the given problem, note first that if n is even, then $n^4 + 4^n$ is even, so composite. Suppose now that n is odd. Then $n = 2m+1$ for some positive integer m. Hence $n^4 + 4^n = n^4 + 4^{2m+1} = n^4 + 4 \cdot 2^{4m} = a^4 + 4b^4$ with $a = n$, $b = 2^m$. Since $b = 2^m \geq 2^1 = 2$, by the above result it follows that the latter expression is composite.

5. (A2, Putnam 1990) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, where n and m are nonnegative integers?

   Solution: The key lies in the observation that the sequence $x_n = n^{1/3}$ is increasing to infinity, with the difference of consecutive terms going to 0. Indeed, by the mean value theorem, we have for some $t \in [n, n+1]$, $x_{n+1} - x_n = (n+1)^{1/3} - n^{1/3} = (1/3)t^{-2/3} \leq (1/3)n^{-2/3}$, which tends to 0 as $n \to \infty$. The claimed result holds for any sequence x_n with this property, and for any real number α in place of $\sqrt{2}$. This is easy to see: Given a number α, which we may assume to be positive without loss of generality, and given an $\epsilon > 0$, choose N so that $0 < x_{n+1} - x_n \leq \epsilon$ for $n \geq N$. Since $x_n \to \infty$, it follows the sequence $x_{N+k} - x_N$, $k = 1, 2, \ldots$, is increasing to infinity, with consecutive terms differing by at most ϵ. Hence, for some k, $x_{N+k} - x_N$ falls into the interval $[\alpha, \alpha + \epsilon]$, and hence $|\alpha - (x_{N+k} - x_N)| \leq \epsilon$.

6. Determine, with proof, the set of positive integers that can be expressed in the form $m = [n + \sqrt{n} + 1/2]$ for some positive integer n.

   Solution: We show that the set in question consists of all positive integers except the squares. Note that a positive integer m is *not* of the required form if and only there is an integer n such that

   $$n + \sqrt{n} + 1/2 < m \quad \text{and} \quad n + 1 + \sqrt{n+1} + 1/2 \geq m + 1.$$

   We rewrite this condition as

   $$\begin{gathered} \sqrt{n} < m - n - 1/2 \leq \sqrt{n+1}, \\ n < (m - n - 1/2)^2 = (m-n)^2 - (m-n) + 1/4 \leq n + 1, \\ n - 1/4 < (m-n)^2 - (m-n) \leq n + 3/4. \end{gathered}$$

   Since $(m-n)^2 - (m-n)$ is an integer, the latter relation holds if and only if $n = (m-n)^2 - (m-n)$, i.e., if $m = (m-n)^2$. Hence the numbers omitted by the sequence $[n + \sqrt{n} + 1/2]$ are exactly the perfect squares.

7. Let $a_n = [(2 + \sqrt{3})^n]$. Show that a_n is odd for every nonnegative integer n.

   Solution: The trick is to relate a_n to the solution of a two term linear recurrence whose parity can easily be determined directly from the recurrence.

To do this, write $a_n = [\alpha^n]$ with $\alpha = 2 + \sqrt{3}$, and note that α is a solution to $(x-2)^2 = 3$ or $(*)$ $x^2 = 4x - 1$. The other solution to the equation $(*)$ is $\beta = 2 - \sqrt{3}$. Now $(*)$ is the characteristic equation for the recurrence $(**)$ $u_n = 4u_{n-1} - u_{n-2}$. By the general theory for linear recurrences, the general solution to $(**)$ is a linear combination of α^n and β^n. In particular $u_n = \alpha^n + \beta^n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$ is a solution, with initial conditions $u_0 = 2$, $u_1 = (2+\sqrt{3}) + (2-\sqrt{3}) = 4$. Since u_0 and u_1 are even integers, and the above recurrence has integer coefficients, it follows by induction that all terms u_n are even integers. Since $0 < 2-\sqrt{3} < 1$, we have $a_n = [(2+\sqrt{3})^n] = [u_n - (2-\sqrt{3})^n] = u_n - 1$. Hence a_n is an odd integer for all n.