Problem Set 1: Binomial identities
Hints and Solutions

1. $\sum_{k=0}^{n} \binom{n}{k}$

   Answer: 2^n (interpret the sum as the number of subsets of an n-element set).

2. $\sum_{k=0}^{n}(-1)^k \binom{n}{k}$

   Answer: 1 if $n = 0$ and 0 if $n \geq 1$ (apply the binomial theorem with $x = -1$).

3. $\sum_{k=0}^{2n}(-1)^k k^n \binom{2n}{k}$

   Answer: 0 if $n \geq 1$. (This is a seemingly impossibly hard problem that, surprisingly, becomes doable (though still anything but routine) in a more general form, namely with k^n replaced by a general power k^r, $r = 0, 1, \dots, 2n-1$. The case $r = 0$ is just the sum in the previous problem; to obtain the general case, one can use induction on r, or compute successive derivatives of $(1 + x)^{2n}$, using the binomial theorem, and then evaluate at $x = -1$. The sum turns out to be 0 for all exponents $r < 2n$.)

4. $\sum_{k=0}^{n} \binom{n}{k}^2$

   Answer: $\binom{2n}{n}$ (write the second binomial as $\binom{n}{n-k}$ and apply Vandermonde's identity (see below)).

5. $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$

   Answer: $\frac{1}{n+1}(2^{n+1} - 1)$ (Integrate the identity $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ from 0 to 1.)

6. $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$

   Answer: $\binom{m+n}{r}$. (This is a famous identity, the **Vandermonde identity**. For a combinatorial proof consider the disjoint union of an m-element and an n-element set and count the number of r-element subsets of this union in two different ways. It can also be proved using generating functions, by considering the r-th coefficient in $(1 + x)^{m+n} = (1 + x)^m(1 + x)^n$.)

7. $\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$

   Hint: The answer is 2^{2n}. Despite the simple form of the answer, this is not an easy problem. One way to do it is via generating functions: Write $\sum_{n=0}^{\infty}(-1)^n x^n = (1 + x)^{-1} = (1 + x)^{-1/2}(1 + x)^{-1/2}$ and expand each of the factors $(1 + x)^{-1/2}$ into a binomial series $\sum_{k=0}^{\infty} \binom{-1/2}{k}$. Comparing coefficients gives $(-1)^n = \sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k}$. Using the relation $\binom{-1/2}{k} = (-1)^k 2^{-2k}\binom{2k}{k}$ then yields an evaluation for the given sum. (Problem: Find a combinatorial proof!)

8. $\sum_{k=0}^{n} \binom{n-k}{k}(-1)^k 2^{-2k}$

   Hint: Let $S(n)$ denote the above sum. A direct computation shows that the first few values of $S(n)$ are $1 = 2/2$, $3/4$, $1/2 = 4/8$, $5/16$, suggesting that $(*)$ $S(n) = (n + 1)/2^n$. This turns out to be the correct answer. To prove $(*)$, first use the recurrence formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ (which holds for any positive integer k and any integer n (positive or negative, with the convention that $\binom{m}{k} = 0$ if $m < k$), to express the sum $S(n)$ in terms of $S(n - 1)$ and $S(n - 2)$, and then prove $(*)$ by induction on n.

9. $\sum_{m=0}^{n} \binom{m}{k}$

   Answer: $\binom{n+1}{k+1}$ (Here k and n are arbitrary nonnegative integers; note that, by definition, $\binom{m}{k} = 0$ when $m < k$. To prove this formula, consider all $(k+1)$-element subsets of the set $\{1, 2, \ldots, n+1\}$, and group them according to their greatest element. The term $\binom{m}{k}$ counts the number of those subsets whose greatest element is $m+1$)

10. $\sum_{k=0}^{n} \frac{\binom{m}{k}}{\binom{n}{k}}$ $(n \geq m)$

    Answer: $\binom{n+1}{m+1}/\binom{n}{m}$. (This one is harder, as it involves quotients instead of products. The trick is to use (and prove) the identity $\binom{m}{k}/\binom{n}{k} = \binom{n-k}{m-k}/\binom{n}{m}$ to reduce the sum to one of the same type as in the previous problem.)

Problem Set 2: Combinatorial problems Hints and Solutions

1. How many subsets are there in a set with n elements?

   Answer: 2^n (see the first problem in Set 1)

2. How many of these subsets have an *even* number of elements?

   Answer: 2^{n-1} if $n \geq 1$ (see the second problem in Set 1, which shows that the number of subsets with an even number of elements is equal to the number of subsets with an odd number of elements).

3. How many ways are there to place an order of 10 donuts if there are 3 varieties to choose from?

   Answer: $\binom{10+3-1}{3-1}$ (Imagine the donuts lined up with two dividers between the different varieties, for a total of 12 spots, two for the dividers, and 10 for the donuts. Then count the number of ways to pick the two spots for the dividers out of the 12 available spots.)

4. In how many ways can n be written as a sum of k nonnegative integers, if the order is taken into account (so that, for example, $10 = 3 + 3 + 4$ and $10 = 3 + 4 + 3$ count as different representations)?

   Answer: This is equivalent to the previous problem with n donuts and k varieties. The answer is $\binom{n+1}{k-1}$.

5. How many ways are there to form a committee of 5 in a group of 20 people?

   Answer: $\binom{20}{5}$

6. How many 14 letter "words" can be formed by rearranging the letters of "FIGHTING ILLINI".

   Answer: $\binom{14}{5}\binom{9}{2}\binom{7}{2}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1}$. or $14!/(5!2!2!1!1!1!)$. (The word contains 5 I's; 2 N's; 2 G's; 2 L; 1 F; 1 H; and 1 T. First place 5 out of the 14 available slots for the letters and place the I's in those slots ($\binom{14}{5}$ ways to do that), then pick 2 out of the remaining 9 free slots for the N's ($\binom{9}{2}$ ways for this), and so on.)