UIUC Putnam Training Sessions
Hints and Solutions

Problem Set 5: Inequalities

1. Given n positive real numbers with sum 1, show that the sum of the squares of these numbers is at least $1/n$.

   Hint: Apply Cauchy to the sum $1 = \sum_{i=1}^n 1 \cdot a_i$.

2. Given n positive real numbers $a_1, \dots, a_n$, define
$$H = n \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} \right)^{-1}.$$
   (The number H is called the **harmonic mean** of the numbers a_i.) Show that $H \leq G$, where $G = (a_1 \dots a_n)^{1/n}$ is the geometric mean of the a_i's.

   Hint: Note that H is the reciprocal of the arithmetic mean of the numbers $b_i = 1/a_i$, while G is the reciprocal of the geometric mean of these numbers. Then apply the Arithmetic-Geometric Mean (AGM) inequality.

3. Let $a_1, \dots, a_n$ be positive integers, and let $b_1, \dots, b_n$ be a permutation of the a_i's. Show that $\sum_{i=1}^n (a_i/b_i) \geq n$.

   Hint: Apply AGM to the numbers a_i/b_i.

4. Suppose f is a nonnegative function defined on the interval $[0, 1]$ and satisfying $\int_0^1 f(x)^2 dx = 1$. What is the maximal value of $\int_0^1 f(x)x^{2002} dx$?

   Hint: Let I denote the integral $\int_0^1 f(x)x^{2002} dx$. Apply the integral version of Cauchy's inequality with the functions $f(x)$ and x^{2002} to get $I^2 \leq \int_0^1 f(x)^2 dx \int_0^1 x^{4004} dx = 1/4005$, or $I \leq 1/\sqrt{4005}$. To show that this upper bound can be achieved, take f to be proportional to x^{2002}, i.e., $f(x) = cx^{2002}$, with $c = \sqrt{4005}$.

5. For s real and positive, define $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$. Prove that $\{\log \Gamma\}'' > 0$ on this half line.

   Hint: First note that $(\log \Gamma)'' = (\Gamma'/\Gamma)' = (\Gamma\Gamma'' - \Gamma'^2)/\Gamma^2$, so the problem is equivalent to showing that $\Gamma\Gamma'' > \Gamma'^2$ on the positive real axis. Differentiating under the integral sign gives $\Gamma'(s) = \int_0^\infty e^{-x} x^{s-1} (\log x) dx$. Now split the integrand into factors $e^{-x/2} x^{(s-1)/2}$ and $e^{-x/2} x^{(s-1)/2} \log x$, and apply Cauchy.

6. Let $x_1, \dots, x_n$ be real numbers with $0 < x_i < 1$, and let $x = (1/n) \sum_{i=1}^n x_i$ be the arithmetic mean of these numbers. Show that
$$\prod_{i=1}^n \left(\frac{\sin x_i}{x_i} \right) \leq \left(\frac{\sin x}{x} \right)^n.$$

   Hint: Taking logarithms the inequality to be shown is equivalent to $\sum_{i=1}^n \log((\sin x_i)/x_i) \leq n \log((\sin x)/x)$. Apply convexity with $f(x) = \log((\sin x)/x)$.

7. Let u, v, w be real numbers. Show that

$$\frac{u+v+w}{3} \le \log \frac{e^u + e^v + e^w}{3}.$$

When does equality hold?

Hint: Exponentiate both sides, then apply the convexity inequality with the function $f(x) = e^x$ to get $f((1/3)u + (1/3)v + (1/3)w) \le (1/3)(f(u) + f(v) + f(w))$.

8. Suppose $x_1, \ldots, x_n$ are positive real numbers with $\sum_{i=1}^n x_i = 1$. Show that

$$\sum_{i=1}^n x_i \log x_i \le \log \sum_{i=1}^n x_i^2.$$

Hint: Apply the convexity inequality with $f(x) = -\log x$ (which is convex since $\log x$ is concave) and $p_i = x_i$.