UIUC Putnam Training Sessions

The Pigeonhole Principle (or Box Principle)

If $n+1$ objects ("pigeons") are distributed among n boxes ("pigeon holes"), at least one of the boxes contains more than one object. More generally, if $kn+1$ objects are distributed among n boxes, at least one of the boxes contains more than k objects.

Problem Set 6: Pigeonhole Problems

1. Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most $1/2$.

2. Show that among any five points inside a 1×1 square there exist two points whose distance is at most $1/\sqrt{2}$.

3. Given a set of 2002 integers show that there exist two of them whose difference is divisible by 2001.

4. Given a set of 7 integers, show that there exist two of them whose difference or sum is divisible by 10.

5. Prove that from a set of ten distinct two-digit integers it is possible to select two disjoint non-empty subsets whose members have the same sum.

6. Show that if n people are at a party, at least two of them know the same number of people (among the other party guests). (Assume that "knowing" is a symmetric relation; that is, if A knows B, then B knows A.)

7. Prove that there exist integers a, b, c, not all zero, and each of absolute value at most 10^6, such that $|a+b\sqrt{2}+c\sqrt{3}| < 10^{-11}$.

8. Let S be the set of real numbers of the form $a+b\sqrt{2}$, where a and b are integers. Show that S is *dense* on the real line, in the sense that, given any $\epsilon > 0$ and any real number x there exists an element $s \in S$ with $|s-x| < \epsilon$.

9. The Fibonacci sequence is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that, given any positive integer k, there exists a Fibonacci number F_n ending in at least k zeros.

10. Show that any set $A \subset \{1, 2, \ldots, 2n\}$ with at least $n+1$ elements contains two elements, one of which divides the other.

11. Suppose $\mathcal{A}$ is a collection of subsets of $\{1, 2, \ldots, n\}$ with the property that any two sets in $\mathcal{A}$ have a non-empty intersection. Show that $\mathcal{A}$ has at most 2^{n-1} elements. Can the bound 2^{n-1} be lowered?

12. (Putnam '95) For a partition π of $\{1, 2, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, \ldots, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. (A partition of a set S is a collection of nonempty subsets (parts) whose union is S.)

13. Prove that any permutation of the integers $1, 2, \ldots, 101$ contains a monotonic (increasing or decreasing) subsequence of 11 terms. (This is a famous result that requires an ingeneous application of the pigeonhole principle that is far from obvious.)