Sequences and Series Toolbox

1 Definitions

You should know the following definitions from 8.1 through 8.5:

- **Sequence** (p.612) A sequence is any function whose

- **Convergent Sequence** (p.613) The sequence $\{a_n\}_{n=n_0}^{\infty}$ converges to L if and only if

- **Factorial** (p.618) For any integer $n \geq 1$, the factorial $n!$ is

- **Increasing sequence** (p.619) The sequence $\{a_n\}_{n=n_0}^{\infty}$ is increasing if

- **Decreasing sequence** (p.619) The sequence $\{a_n\}_{n=n_0}^{\infty}$ is decreasing if

  If a sequence is either increasing or decreasing it is called ***monotonic****.*

- **Bounded sequence** (p.620) We say that the sequence $\{a_n\}_{n=n_0}^{\infty}$ is bounded if

- **Infinite series** (p.627) $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty}$

- **Geomentric series** (p.629) For $a \neq 0$,

  What goes wrong if $a = 0$?

- **Harmonic series** (p.632)

  What is the harmonic series famous for?

- P**-series** (p.63__)

  When does a p-series converge?

- **Alternating series**

  How is the notation a_k in an alternating series different from a_k in other series?

- **Absolute convergence** (p.656) If $\sum_{k=1}^{\infty}$ is convergent, we say $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

- **Conditional convergence** (p.656)

2 Theorems

These theorems are useful for determining if a *sequence* converges.

- If $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ both , then

  1. $\lim_{n\to\infty}(a_n + b_n) =$
  2. $\lim_{n\to\infty}(a_n = b_n) =$
  3. $\lim_{n\to\infty}(a_n b_n) =$
     is this true if $\lim_{n\to\infty} a_n = \infty$*?*
  4. $\lim_{n\to\infty} \frac{a_n}{b_n} =$
     (assuming $\lim_{n\to\infty} b_n \neq$ ____.)

- **The Theorem**

  Suppose $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ are convergent sequences, both converging to L. If there exists an integer $n \geq n_0$ such that for all n bigger than n_0 , then $\{c_n\}_{n=n_0}^{\infty}$ converges to .

- If $\lim_{n\to\infty} |a_n| =$ ____, then $\lim_{n\to\infty} a_n =$ ____.

  Is this true for any limit?

- If an increasing sequence is , it converges. If a decreasing sequence is , it converges.

  Notice that if a monotonic sequence (that is, either increasing or decreasing) is ***bounded*** *(that is, bounded above as well as below), then it converges.*

  If an increasing sequence is bounded below, must it necessarily diverge?

- If $\{a_n\}_{n=n_0}^{\infty}$ is a convergent sequence with limit L, then any infinite subsequence $\{a_{n_k}\}_{k=k_0}^{\infty}$ also converges to L.

  How can you use this to prove that $a_n = (-1)^n \frac{n}{3n+2}$ *is a divergent sequence?*

- If the *function* $f(x)$ converges to L as $x \to \infty$, then the *sequence* $a_n = f(n)$.

  If $\lim_{n\to\infty} a_n = L$*, does* $\lim_{x\to\infty} f(x) = L$*?*

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then the sequence $\{a_n\}_{n=n_0}^{\infty}$ is " eventually ."

The following theorems are useful for determining if a series converges. They are listed from easiest to use to hardest. Always make sure your series satisfies all of the hypotheses, and that you know what the conclusion really says!

- If $\sum_{k=1}^{\infty} a_k$ converges to A and $\sum_{k=1}^{\infty} b_k$ converges to B, then the series $\sum_{k=1}^{\infty}(a_k \pm b_k)$ converges to ________ and $\sum_{k=1}^{\infty} ca_k$ converges to ____ for any constant c.

- If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then the series $\sum_{k=1}^{\infty}(a_k \pm b_k)$ ____________________.

- **The geometric series** $\sum_{k=1}^{\infty} a(r)^k$ converges to if , and diverges otherwise.

  Hypotheses: The series is a geometric series with .

  Conclusion: The series converges to .

- **The kth term test for divergence**.

  If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

  The contrapositive is most useful: if $\lim_{k\to\infty} a_k \neq 0$ *then* $\sum_{k=1}^{\infty} a_k$ ***diverges***.

  Hypotheses: $\lim_{k\to\infty} a_k \neq 0$

  Conclusion: $\sum_{k=1}^{\infty} a_k$ **diverges**

- **p-series** The p-series $\sum_{k=1} \frac{1}{k^p}$ converges if and diverges if .

  Hypotheses:

  Conclusion:

- **Alternating series test**. If $\lim_{k\to\infty} a_k = 0$, and $0 < a_{k+1} < a_k$ for all $k \geq 1$, then the alternating series $\sum_{k=1}^{\infty}$ converges.

  Hypotheses:

  Conclusion:

  Don't get carried away and try to use this test on a non-alternating series!

- **Integral test** If $f(k) = a_k$ for all $k = 1, 2, 3,...$ and f is continuous and decreasing, and $f(x) \geq 0$ for $x \geq 1$, then $\int_1^{\infty} f(x)dx$ and $\sum_{k=1}^{\infty} a_k$ either *both converge* or *both diverge*.

  Hypotheses:

  Conclusion:

  Think about when you can't use this test and why. Come up with some examples to help you remember.

- **Comparison test** Suppose that $0 \leq a_k \leq b_k$ for all k.
  1. If $\sum_{k=1}^{\infty} b_k$ converges, then
  2. If $\sum_{k=1}^{\infty} a_k$ diverges, then

  Think about why we require $0 \leq a_k$.

- **Limit comparison test** Suppose $a_k, b_k > 0$ and that for some finite value L, $\lim_{k\to\infty} \frac{a_k}{b_k} = L >$ ____. Then

  Hypotheses:

  Conclusion:

  If your ***comparison test*** *fails, try a* ***limit comparison test*** *instead with the same* b_k.

- **Ratio test** Given $\sum_{k=1}^{\infty} a_k$, with $a_k \neq$ ____ for all k, suppose that

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L$$

  Then
  1. If $L < 1$
  2. If $L = 1$
  3. If $L > 1$

Hypotheses:

Conclusion:

Warning: sometimes this test gives you no information at all! This sort of test is easiest to do when a_k involves things that cancel nicely in the ratio, such as 3^n and $n!$.

- **Root test** Given $\sum_{k=1}^{\infty} a_k$, suppose that $\lim_{k\to\infty}$ $= L$. Then.

  1. If $L < 1$
  2. If $L = 1$
  3. If $L > 1$

  Hypotheses:

  Conclusion:

  Warning: sometimes this test gives you no information at all! This test is easiest to do when a_k is all raised to the power k.