1 Sequences and Series

Infinite Series: A concept related to the idea of a sequence is an (infinite) series. A series is an infinite sum of the form

$$\sum_{n=1}^{\infty} a_n$$

We say that a series converges if the limit

$$\lim_{N \to \infty} \sum_{n=1}^{N} a_n$$

exists (This is the sequence of partial sums). Note that this is **exactly** analogous to improper integrals, where we say that the integral converges if the limit

$$\lim_{R \to \infty} \int_1^R f(x)dx$$

converges.

Examples

$$\textstyle\sum_{n=1}^{\infty} ar^n \tag{1}$$

$$\textstyle\sum_{n=1}^{\infty} \frac{1}{n} \tag{2}$$

$$\textstyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \tag{3}$$

Example

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

Converges. **Proof:**

Note that

$$\begin{aligned} n^2 &> n(n-1) \\ \frac{1}{n^2} &< \frac{1}{n(n-1)} \end{aligned}$$

$$S_N = \sum_{n=2}^{N} \frac{1}{n^2} < \sum_{n=2}^{N} \frac{1}{n-1} - \frac{1}{n} = 1 - \frac{1}{N} < 1.$$

S_N is increasing and bounded, so the series converges.

Theorem:

If $\lim_{n \to \infty} a_n \neq 0$ then

$$\sum_{n=1}^{\infty} a_n$$

does not converge. **Proof:** Define the partial sums

$$S_N = \sum_{n=1}^{N} a_n$$

Note that

$$\lim_{N\to\infty} S_N = L$$

As well as

$$\lim_{N\to\infty} S_{N-1} = L$$

Thus

$$\lim_{N\to\infty} S_N - S_{N-1} = 0$$

NOTE: $\lim_{n\to\infty} a_n = 0$ does **NOT** guarantee that

$$\sum_{n=1}^{\infty} a_n$$

converges!

Example:

$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$$

diverges, since

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$$

IMPORTANT EXAMPLE: HARMONIC SERIES

Claim: The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

DIVERGES!

To begin with we look at a few partial sums:

$$\begin{aligned} S_{10} &= 2.93 \\ S_{100} &= 5.19 \\ S_{1000} &= 7.49 \\ S_{10000} &= 9.79 \\ S_{100000} &= 12.09 \end{aligned}$$

Numerically it looks like each time N gets multiplies by a factor of 10 the sum increases by about 2.3 This is actually true, with the number 2.3 being $\log(10) \approx 2.3026$.

Proof: We are going to relate it to another sum which diverges.

$$\begin{aligned}
\sum_{n=1}^{2^k-1} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \frac{1}{2^k} \\
&= 1 + \underbrace{\frac{1}{2} + \frac{1}{3}} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}} + \underbrace{\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}} + \dots \underbrace{\frac{1}{2^{k-1}} + \frac{1}{2^{k-1}+1}} \\
&\geq 1 + 2(\frac{1}{3}) + 4(\frac{1}{7}) + 8(\frac{1}{15}) + \dots \\
&\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots
\end{aligned}$$

So adding 2^k more terms increases the sum by 1/2. Thus the sum cannot approach a limit, since I can make the partial sum increase arbitrarily much.

Aside: It can be shown that

$$\sum_{n=1}^{N} \frac{1}{n} \approx \ln(N) + \gamma$$

where γ is a constant known as the Euler-Mascheroni constant. This follows from something called the integral test, which is the subject of Section 8.3.

Big Theorem: Integral Test

Theorem: Suppose that $f(x) >= 0$ is a continuous decreasing function and $a_n = f(n)$. The either $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ both **Converge**, or they both **Diverge.**

Proof: This is basically a proof by pictures. Suppose that $f(x)$ is decreasing and positive. Then we have the inequalities

$$\sum_{2}^{N} a_n \leq \int_1^N f(x)dx \leq \sum_{1}^{N-1} a_n$$

If the improper integral converges then

$$\begin{aligned}
\sum_{2}^{N} a_n &\leq \int_1^{\infty} f(x)dx \\
\sum_{1}^{N} a_n &\leq a_1 + \int_1^{\infty} f(x)dx
\end{aligned}$$

So the partial sums are increasing and bounded, and thus converge.

Similarly if the sum converges then

$$\int_1^R f(x)dx \sum_{1}^{\infty} a_n$$

so $\int_1^R f(x)dx$ is a bounded increasing function, and thus converges.

Example: Harmonic Series Take $f(x)$ to be

$$f(x) = \frac{1}{x}.$$

This is a continuous decreasing function on $[1, \infty)$. Thus the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

and the (improper) integral

$$\int_1^{\infty} \frac{dx}{x}$$

either **both** converge, or they **both** diverge. The improper integral is easy to evaluate

$$\int_1^R \frac{dx}{x} = \ln |x||_1^R = \ln |R| - \ln |1| = \ln |R|$$

as $R \to \infty$ the righthand side goes to infinity. Hence the improper integral diverges, and thus the sum diverges.