1 Convergence of Series

Last time we introduced the first important test: the integral test. The integral test says the following:

Theorem: Integral Test Suppose $f(x)$ is continuous function which is eventually decreasing and positive, and $a_n = f(n)$. Then either

$$\sum_1^\infty a_n$$
$$\int_1^\infty f(x)dx$$

both converge, or they both diverge.

This is a convenient test because it ALWAYS gives an definite outcome (assuming that one can test convergence of **ONE** of the series. Let's look at some examples:

Example: The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Converges if $p > 1$ and diverges if $p \leq 1$. This is easy to see by applying the integral test. It is easy to see that $f(x) = \frac{1}{x^p}$ is decreasing for $p > 0$, and is continuous on $[1, \infty)$. Computing the integral

$$\lim_{R\to\infty} \int_1^R \frac{dx}{x^p} = \lim_{R\to\infty} \frac{x^{1-p}}{1-p}|_1^R \quad (1)$$

$$= \lim_{r\to\infty} \frac{1}{1-p}(1 - R^{1-p}) = \begin{cases} 0 & p > 1 \\ \infty & p < 1 \end{cases} \quad (2)$$

In the case $p = 1$ one gets

$$\lim_{R\to\infty} \int_1^R \frac{dx}{x} = \lim_{R\to\infty} \ln(x)|_1^R = \lim_{R\to\infty} \ln(R) = \infty$$

Aside: The function

$$f(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called the Riemann Zeta function. There is a conjecture (the Riemann hypothesis) that says that $f(p) = 0$ only if $p = a + i/2$. If you prove this you win a million bucks.

Example

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + 1}$$

Converges. **Proof:** The integral

$$\int_0^R \frac{dx}{1 + x^2} = \arctan(x)|_1^R = \arctan(R) - \frac{\pi}{4}$$

converges since $\lim_{R\to\infty} \arctan(R) = \frac{\pi}{2}$.

Example:

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

We apply the integral test with the function $f(x) = xe^{-x}$. We know that $\int_1^{\infty} xe^{-x}dx$ converges, so the sum converges.

Theorem: Comparison Test

If $0 \geq a_n \geq b_n$ then

- $\sum b_n$ converges implies $\sum a_n$ converges
- $\sum a_n$ diverges implies $\sum b_n$ diverges

Note: This is EXACTLY like the comparison test for integrals.

Example

$$\sum \frac{1}{44n^2 + 81n + 273}$$

converges since

$$\begin{aligned} 44n^2 + 81n + 273 &> 44n^2 && (3) \\ \frac{1}{44n^2 + 81n + 273} &< \frac{1}{44n^2} && (4) \end{aligned}$$

Example

$$\sum \frac{1}{n + \sqrt{(n)}}$$

DIVERGES since

$$\begin{aligned} n &\geq \sqrt{n} && (5) \\ 2n &\geq n + \sqrt{n} && (6) \\ \frac{1}{2n} &\leq \frac{1}{n + \sqrt{n}} && (7) \end{aligned}$$

So by comparison to the series

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

the series

$$\sum \frac{1}{n + \sqrt{(n)}}$$

is divergent.

The comparison test is useful, but it is sometimes hard to use because the inequalities have to go the correct way in order to get any information. There is a better version of the comparison which I find a lot easier to use in practice. This is the limit comparison test:

Limit Comparison Test: Suppose that for n large enough $a_n, b_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

Then either both

$$\sum a_n$$
$$\sum b_n$$

CONVERGE or they both DIVERGE.

Example:

$$\sum_{n=1}^{\infty} \frac{n^2 + 2e^{-n} + 88n\cos(n) + 1}{n^4 + n^3 \sin(n) + e^{-2n} - 5}$$

CONVERGES. To see this note that when n is large

$$\frac{n^2 + 2e^{-n} + 88n\cos(n) + 1}{n^5 + n^3 \sin(n) + e^{-2n} - 5} \approx \frac{n^2}{n^5} = \frac{1}{n^3}$$

Letting $b_n = \frac{1}{n^3}$ and $a_n = \frac{n^2+2e^{-n}+88n\cos(n)+1}{n^5+n^3\sin(n)+e^{-2n}-5} \approx \frac{n^2}{n^5}$ we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 + 2n^{-2}e^{-n} + 88n^{-1}\cos(n)}{1 + n^{-2}\sin(n) + n^{-5}e^{-n} - 5n^{-5}} = 1$$

Thus we know that either both converge or both diverge. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges we know that both converge.

Rate of Convergence

It is not always enough to know that a series converges. Often one would like to get some estimate of the rate of convergence. In particular one would like to understand how many terms are necessary to estimate the sum to a particular accuracy. One particularly useful fact is the following estimate: if $f(x)$ is positive, decreasing

$$\sum_{n+1}^{\infty} f(n) \le \int_{n}^{\infty} f(x)dx$$

This is best illustrated with an example:

Example: How many terms N are necessary so that $\sum_{n=1}^{N} \frac{1}{n^2}$ is guaranteed to be within 10^{-3} of the actual sum? Within 10^{-8}?

Solution: $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Then

$$|S - \sum_{n=1}^{\infty} \frac{1}{n^2}| = |\sum_{N+1}^{\infty} \frac{1}{n^2}| \leq \int_N^{\infty} \frac{dx}{x^2} = -\frac{1}{x}|_N^{\infty} = \frac{1}{N}$$

So in the first case we need $\frac{1}{N} < 10^{-3}$ or $N > 10^3$. IN the second case we need $\frac{1}{N} < 10^{-8}$ or $N > 10^8$.

Example: How many terms are necessary to estimate

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1}$$

to within 10^{-4}?

Solution: Again we have

$$|S - \sum_{n=1}^{\infty} \frac{1}{n^3+1}| = |\sum_{N+1}^{\infty} \frac{1}{n^3+1}| \leq \int_N^{\infty} \frac{dx}{x^3+1}$$

But the integral is no longer so simple (though it can be done - think about how you would do it). An easier way to proceed is to notice that

$$\frac{1}{x^3+1} \leq \frac{1}{x^3}$$

and thus

$$\int_N^{\infty} \frac{dx}{x^3+1} \leq \int_N^{\infty} \frac{dx}{x^3} = -\frac{1}{2}x^{-2}|_N^{\infty} = \frac{-1}{2N^2}$$

so we require that

$$\tfrac{1}{2N^2} \leq 10^{-4} \tag{8}$$

$$2N^2 \geq 10^4 \tag{9}$$

$$N \geq \sqrt{5 \times 10^3} \approx 71 \tag{10}$$

Note: The series $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3+1}$ both converge, but the latter converges much faster. The first series you need 1000 terms to estimate the answer to within .001. The second you need 70 terms to estimate the answer within .0001 !