

1 Alternating Series

Last time we talked about convergence of series. Our two most important tests, the integral test and the (limit) comparison test, both required that the terms of the series were (eventually) positive. There is an important role played by series with alternating terms. It is, in a sense, much easier for series with terms of alternating terms to converge since there is “cancellation” between the terms of the series.

Lets begin with a numerical example. I'd like to consider two series

$$S = \sum_{k=1}^{\infty} \frac{1}{k}$$
$$S' = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

As always we let S_N, S'_N denote the partial sums of the series. Let's look at some partial sums:

N	S_N	S'_N
1	1	1
10	2.92	0.646
21	3.65	0.716
32	4.06	0.678
43	4.35	0.704
54	4.58	0.684
65	4.76	0.701

The partial sums of the harmonic series are not obviously converging. In fact we know that the harmonic series does not converge. The partial sums of the *alternating* harmonic series look like they *may* be converging. The hover around .7. I'd also like to note two facts, which will prove important later: The odd partial sums form a decreasing sequence while the even partial sums form an increasing sequence. Further the **largest** term is S_1 , and *every* odd term is larger than every even term. These facts will be true in general, and will let prove that (under certain conditions) alternating series generally converge.

FYI: The sum of the alternating series above is

$$\sum \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \log(2)$$

Now let us state the main theorem in this section:

Alternating Series Theorem:

Suppose that a_k satisfies

- $a_k \rightarrow 0$ as $k \rightarrow \infty$

- $0 < a_{k+1} \leq a_k$ for k large enough.

Then the series

$$\sum -1^{k+1} a_k$$

converges.

Proof: The basis of the proof is to note a couple of facts about the partial sums

$$S_N = \sum_{k=1}^N (-1)^{k+1} a_k$$

Fact 1: The odd terms form an decreasing sequence, the even terms form an increasing sequence.

Note that

$$S_1 = a_1 \tag{1}$$

$$S_2 = a_1 - a_2 = S_1 - a_2 < S_1 \tag{2}$$

$$S_1 > S_3 = a_1 - a_2 + a_3 > S_2 \tag{3}$$

$$S_2 < S_4 = a_1 - a_2 + a_3 - a_4 < S_3 \tag{4}$$

So the odd terms form a decreasing sequence, the even terms an increasing one, with all odd terms larger than all even terms.

This immediately implies that the sequence of *odd* partial sums and the sequence of *even* sums are both convergent! The odd sums are decreasing and bounded below (By any even term), and the even terms are increasing and bounded above (by any odd term). Thus both of these sequences converge.

$$\lim_{N \rightarrow \infty} S_{2N+1} = L_{odd} \tag{5}$$

$$\lim_{N \rightarrow \infty} S_{2N} = L_{even} \tag{6}$$

We don't immediately know that $L_{odd} = L_{even}$ but this is not hard to show. Using the linearity of limits we can see that

$$\lim_{N \rightarrow \infty} (S_{2N+1} - S_{2N}) = \lim_{N \rightarrow \infty} a_{2N+1} = 0 \tag{7}$$

$$= L_{odd} - L_{even} \tag{8}$$

Thus the limits of the even and odd subsequences of partial sums are equal, and the series must converge.

Example: Determine convergence of the following series:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} (k+8)}{k(k+11)}$$

Taking $a_k = \frac{k+8}{k(k+11)}$ we would like to apply our theorem on alternating series. We need to check that a_k is eventually positive and decreasing, and that $a_k \rightarrow 0$. The first is obvious, as is the third. To check the second we need

$$\begin{aligned} a_k - a_{k+1} &= \frac{k+8}{k(k+11)} - \frac{k+9}{(k+1)(k+12)} \\ &= \frac{(k+1)(k+12)(k+8) - k(k+9)(k+11)}{k(k+1)(k+11)(k+12)} \\ &= \frac{k^2 + 17k + 96}{k(k+1)(k+11)(k+12)} > 0 \end{aligned}$$

so the sequence a_k is decreasing. Thus the theorem applies and the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k+8)}{k(k+11)}$$

is convergent.

Example:

Assess the convergence of the series

$$\sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{\ln(k)}$$

Again we take $a_k = \frac{1}{\ln(k)}$. It is clear that the sequence is positive and that $a_k \rightarrow 0$. Checking the difference

$$a_k - a_{k+1} = \frac{1}{\ln(k)} - \frac{1}{\ln(k+1)} = \frac{\ln(k+1) - \ln(k)}{\ln(k)\ln(k+1)} = \frac{\ln \frac{k+1}{k}}{\ln k \ln k + 1} > 0$$

so the sequence is decreasing. Thus the theorem on alternating series applies and the series is convergent.

Let a_k be defined by

$$a_k = \begin{cases} \frac{1}{k} & k \text{ even} \\ \frac{1}{2k} & k \text{ odd} \end{cases}$$

Consider the convergence of the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

The sequence a_k is obviously going to zero and positive. Is it decreasing? Checking we find that if k is even then

$$a_k - a_{k+1} = \frac{1}{k} - \frac{1}{2k+2} = \frac{k+2}{2k(k+1)}$$

while if k is odd we have

$$a_k - a_{k+1} = \frac{1}{2k} - \frac{1}{k+1} = \frac{1-2k}{2k(k+1)}$$

So the sequence is not decreasing. Thus the alternating series test does not give a conclusion. (The series is actually **divergent** though we must do more work to show this.)

Error Estimates

As with other series it is important to have a method for estimating the difference between the value of the series and a partial sum. This is given by

Error Estimate

The error incurred in approximating the series by the first N terms satisfies

$$|S_n - S| < a_{n+1}$$

Proof: Obvious Picture

Example: Find the number of terms necessary to guarantee that the series considered thusfar are approximated to within 10^{-3}

(1) The error in approximating

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k+8)}{k(k+11)}$$

by

$$\sum_{k=1}^N \frac{(-1)^{k+1}(k+8)}{k(k+11)}$$

is less than

$$|S - S_N| \leq \frac{(N+8)}{N(N+11)} \leq \frac{1}{N}$$

Thus to estimate this sum to within 10^{-3} we need to take more than 10^3 terms. This would be extremely quick on any computer.

(2) The error in approximating

$$\sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{\ln(k)}$$

by

$$\sum_{k=3}^N \frac{(-1)^{k+1}}{\ln(k)}$$

is less than $\frac{1}{\ln(N+1)}$. If we demand $\frac{1}{\ln(N+1)} < 10^{-3}$ this is the same as $\ln(N+1) > 10^3$ or $N > e^{10^3} + 1 \approx 1.97 \times 10^{434}$. Compare this to the age of the Universe (10^{17} s) and we see that it would be difficult to sum this series with any accuracy. If we can sum a billion (10^9) terms a second for the lifetime of the universe (10^{17} seconds) our error would be guaranteed to be less than $\frac{1}{\ln(10^{23})} \approx \frac{1}{53} \approx .02$